# Mathematical Tables
## *and other*
# Aids to Computation

A Quarterly Journal

Edited by


E. W. CANNON
C. C. CRAIG
A. ERDÉLYI
F. J. MURRAY
J. TODD
D. H. LEHMER, *Chairman*



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DIGITAL DIFFERENTIAL ANALYZER
CRC 101

# The Difference Analyzer: A Simple Differential Equation Solver

**Introduction.** The difference analyzer is a new type of analog computing machine which permits the solution of sets of non-linear as well as linear ordinary differential equations by a simplified stepwise integration method.

The analog computer solution of a typical set of two ordinary differential equations

$$\frac{dx}{dt} = \psi_1(x, y, t) \qquad \frac{dy}{dt} = \psi_2(x, y, t) \tag{1}$$

is discussed.

In the electromechanical analog computers under consideration here, two voltages

$$\begin{aligned} \frac{dX}{d\tau} &= \frac{a_x}{a_t}\psi_1\left[\frac{1}{a_x}X, \frac{1}{a_y}Y, \frac{1}{a_t}\tau\right] \\ \frac{dY}{d\tau} &= \frac{a_y}{a_t}\psi_2\left[\frac{1}{a_x}X, \frac{1}{a_y}Y, \frac{1}{a_t}\tau\right] \end{aligned} \tag{2}$$

are obtained as functions of the shaft rotations $X$, $Y$, and $\tau$ proportional to $x$, $y$, and $t$, respectively, through the use of universal function potentiometers[1] and resistive summing networks. The scale factors $a_x$, $a_y$, and $a_t$ must be chosen so that neither the shaft rotations nor the voltages (2) exceed their physically possible operating ranges.

At the beginning of each computation, the $\tau$, $X$, and $Y$ shafts are set to initial settings corresponding to the given initial values of $t$, $x$, and $y$; usually $\tau = 0$ at the beginning of a computer run. If, now, the machine can be made to establish the additional relations

$$X = \int_0^\tau \frac{dX}{d\tau}\,d\tau, \qquad Y = \int_0^\tau \frac{dY}{d\tau}\,d\tau \tag{3}$$

between the shaft rotations $X$, $Y$, and the voltages $\frac{dX}{d\tau}$, $\frac{dY}{d\tau}$, then *all these quantities must vary in the manner prescribed by the differential equations (1) and may be recorded as solutions of the problem.*

In machines of the electromechanical differential analyzer type, the integrations (3) are performed continuously, for instance by means of rate servos whose output speed must be accurately proportional to an input voltage. The construction of the electromechanical integrators requires great care and constitutes a major portion of the cost of a differential analyzer.

In the difference analyzer, the integrations are performed by stepwise displacements of the shafts corresponding to the variables in question. The new machine permits rapid, reasonably accurate (one to three percent) solutions at minimum expense, since no servomechanisms or amplifiers need be used. Various applications and refinements of the difference analyzer are also discussed.

**The Difference Analyzer.** In the difference analyzer, a new analog computer first described by the writer,[2] some of the errors and particularly the high costs inherent in the electromechanical integrators are avoided at the expense of some additional computing time.

Fig. 1 illustrates the principle of the difference analyzer. The desired initial values of the variables $\tau$, $X$, and $Y$, are set up as shaft rotations, as before. Then these shaft rotations are varied *step by step* by amounts $\Delta\tau$, $\Delta X$, and $\Delta Y$, respectively, in a manner prescribed by the given mathematical relations (1). The voltages proportional to the derivatives (2) are obtained as functions of the shaft rotations by means of resistive networks and function potentiometers in the manner indicated in Fig. 2.

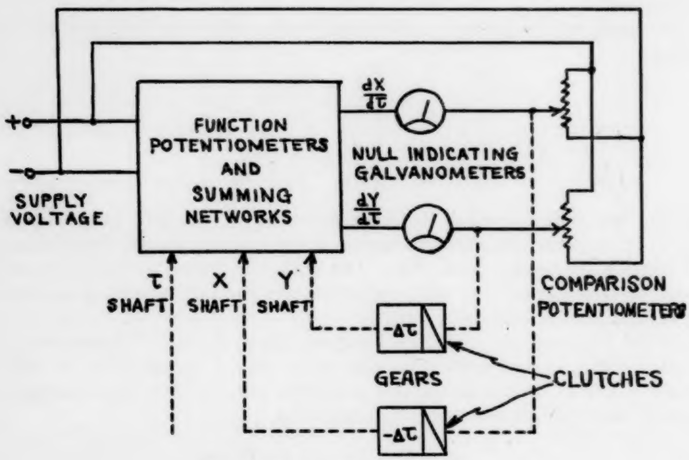


FIG. 1. Simple difference analyzer.

The values of the derivatives $\dfrac{dX}{d\tau}$ and $\dfrac{dY}{d\tau}$ are measured accurately by means of *linear comparison potentiometers* connected to the same supply voltage as the computing networks used to compute $\dfrac{dX}{d\tau}$ and $\dfrac{dY}{d\tau}$. If the scale factors $a_x$, $a_y$, and $a_t$ have been chosen correctly, the displacements of the comparison potentiometers will be equal to $\dfrac{dX}{d\tau}$ and $\dfrac{dY}{d\tau}$, respectively, when the galvanometers shown in Fig. 1 indicate zero current.

The increments $\Delta X$ and $\Delta Y$ of the dependent machine variables, corresponding to a small increment $\Delta\tau$ of the independent machine variable $\tau$ will be approximately

$$\Delta X = \frac{dX}{d\tau}\Delta\tau, \qquad \Delta Y = \frac{dY}{d\tau}\Delta\tau. \tag{4}$$

The multiplications in (4) may be performed simply by means of step-down gears which multiply the respective displacements of the comparison poten-

tiometer shafts by the constant parameter $\Delta\tau$. The resulting shaft rotations $\Delta X$ and $\Delta Y$ are taken as the increments corresponding to an increment $\Delta\tau$ in the displacement of the $\tau$ shaft.

The actual computation proceeds by the iteration of the following two very simple steps.

(i) *Computation of Correct Derivatives.* With the clutches shown in Fig. 1 disengaged, the operator balances the galvanometers by adjusting the comparison potentiometers. The respective displacements of the two potentiometers are now proportional to $\frac{dX}{d\tau}$ and $\frac{dY}{d\tau}$. Note that these settings do *not* have to be read by the operator.

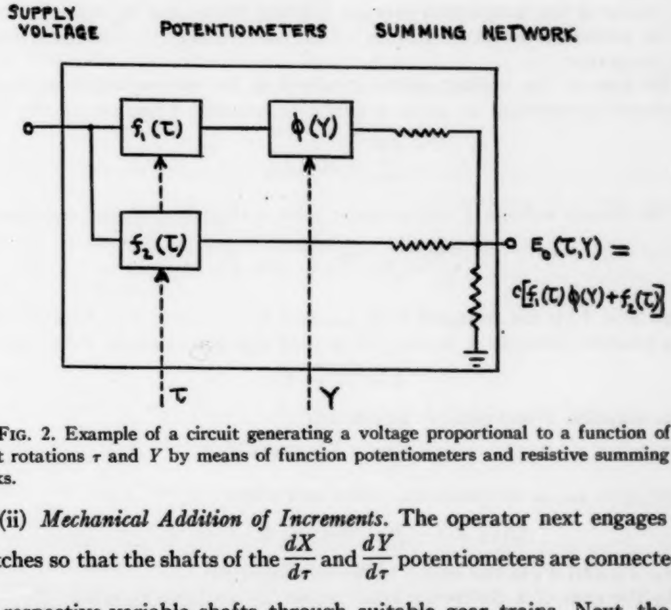


FIG. 2. Example of a circuit generating a voltage proportional to a function of two shaft rotations $\tau$ and $Y$ by means of function potentiometers and resistive summing networks.

(ii) *Mechanical Addition of Increments.* The operator next engages the clutches so that the shafts of the $\frac{dX}{d\tau}$ and $\frac{dY}{d\tau}$ potentiometers are connected to the respective variable shafts through suitable gear trains. Next, the $\frac{dX}{d\tau}$ potentiometer is reset to its center, or zero position. This involves a displacement proportional to $\frac{dX}{d\tau}$ which, with the clutch engaged, is imparted to the $X$ shaft through a gear ratio $1:\Delta\tau$ in such a manner that the correct increment

$$\Delta X = \frac{dX}{d\tau}\Delta\tau$$

is added to the displacement $X$. Similarly, an increment $\Delta Y$ corresponding to (4) is added to the displacement of the $Y$ shaft when the $\frac{dY}{d\tau}$ potentiometer is reset to zero.

The $\tau$ (independent variable) shaft is simply advanced by a specified constant amount $\Delta\tau$.

The operator then disengages the clutches and the whole process is repeated for the newly obtained values of $\tau$, $X$, and $Y$. It is readily seen that the dependent variables $X$ and $Y$ must vary according to the prescribed equations (4); the process just described constitutes a *stepwise integration* of the differential equations (1).

It may be shown that the error involved in this stepwise integration *as such* could be made as small as desired by using sufficiently small time increments $\Delta\tau$.[3] Actually, the accuracy of the machine is determined by the accuracy of the dial settings and by the tolerances of the resistive computing networks. Random errors in dial settings or potentiometer linearity tend to cancel in the course of the integration process. Certain errors due to imperfect computing networks, however, may be integrated and may thus increase during the integration.

Because of the various errors involved in the computation, an analog computer attempting to solve a single differential equation of the form

$$\frac{dY}{d\tau} = f(\tau, Y) \tag{5}$$

for the correct voltage $Y$ will actually solve a slightly different equation

$$\frac{d\bar{Y}}{d\tau} = f(\tau, \bar{Y}) + A(\tau, \bar{Y}),$$

where $A(\tau, \bar{Y})$ is the *derivative error* assumed to be bounded in absolute value by a positive quantity $\delta$. It may be shown[3] that the absolute value error

$$\epsilon = |Y - \bar{Y}|$$

in the solution $Y$ will satisfy the relation

$$\epsilon \leqslant \delta|\tau - \tau_0|e^{M|\tau-\tau_0|}, \tag{6}$$

where $|\tau - \tau_0|$ is the computing time and where

$$|f(\tau_1, Y_1) - f(\tau_1, Y_2)| \leqslant M|Y_1 - Y_2|$$

for $\tau_1$, $Y_1$, and $Y_2$ in the region under consideration.

In the case of a difference analyzer set up to solve equation (5), it is possible to write

$$\delta \leqslant \lambda_1 + \mu_1 + \mu_2\Delta\tau. \tag{7}$$

Here $\lambda_1$ is the absolute value of the greatest possible error in $\left.\frac{d\bar{Y}}{d\tau}\right]_{\tau,\bar{Y}}$ as compared to $f(\tau, \bar{Y})$ due to the computing elements used to produce $f(\tau, \bar{Y})$; $\mu_1$ is the absolute value of the largest possible derivative error incurred in setting the machine variable $\bar{Y}$ to a new value. $\mu_2\Delta\tau$ is a measure of the largest derivative error incurred by replacing $f(\tau, \bar{Y})$ by the constant value of $f(\tau_i, \bar{Y}_i)$ corresponding to $\tau = \tau_i$ and $\bar{Y} = \bar{Y}_i$ in the $i$th step of the stepwise integration. The relations (6) and (7) constitute an estimate of the upper bound of the error made by a difference analyzer in the solution of the given differential equation (5).

**Example of An Error Estimate.** Consider the solution of the machine equation $\frac{dY}{d\tau} = Y$ with $Y = 1$ for $\tau = \tau_0 = 0$ in the interval $0 \leqslant \tau \leqslant \tau_1$. This is a typical problem unfavorable for analog computer solution because of its "unstable" monotonically increasing solution. Here $M = 1$.

If this problem is solved on a typical *difference analyzer* with

$$\lambda_1 \leqslant 0.005 \left.\frac{dY}{d\tau}\right]_{\mathrm{MAX}} = 0.005e^{\tau_1}, \mu_1 \leqslant 0.002 \left.\frac{dY}{d\tau}\right]_{\mathrm{MAX}} = 0.002e^{\tau_1}$$

one has

$$\mu_2 \leqslant \left.\frac{d^2Y}{d\tau^2}\right]_{\mathrm{MAX}} = e^{\tau_1}.$$

In this case, the difference analyzer will be as accurate as a typical differential analyzer with $\delta \leqslant 0.01\, e^{\tau_1}$ if one chooses

$$\Delta\tau \leqslant 0.003.$$

For large values of $\tau_1$, it might pay in this particular problem to divide the interval under consideration into subintervals, all but one of which would yield smaller values of $\mu_2$.

**Practical Results.** A small difference analyzer was used by the writer to solve Euler's equations of motion for a rigid body.[4] As a test problem, the system of differential equations

$$\frac{dX}{d\tau} = Y, \qquad \frac{dY}{d\tau} = -X \tag{8}$$

with $X = 0$ and $Y = 1$ for $\tau = 0$, which has the well known solutions

$$X = \sin\tau \text{ and } Y = \cos\tau$$

was solved on the difference analyzer. A computer setup for the system (8) is shown in Fig. 3.

The construction of resistive networks with an accuracy of better than 0.2 percent is well known to the art and quite easily possible with relatively low-cost components. Helical potentiometers were used in the setup of Fig. 3. For a 45-step integration from $\tau = 0$ to $\tau = \pi/4$, better than one percent overall accuracy in terms of full scale readings was attained at less than one-third the cost of a comparable differential analyzer.

**Refinements.** The computing networks of a difference analyzer may be interconnected by means of patchcords so that a *flexible multi-purpose computer* results.

The advantages inherent in the low cost of the difference analyzer become apparent particularly when non-linear differential equations are to be solved. Again, higher order differential equations can always be reduced to systems of first-order equations by introducing the derivatives as new variables.

The possibility of introducing new variables simply by means of new dependent variable shafts in the difference analyzer suggests the use of such "mechanical transformations" to simplify complicated computer setups and to improve scale factors with resulting increases in accuracy. Fig. 4 shows how

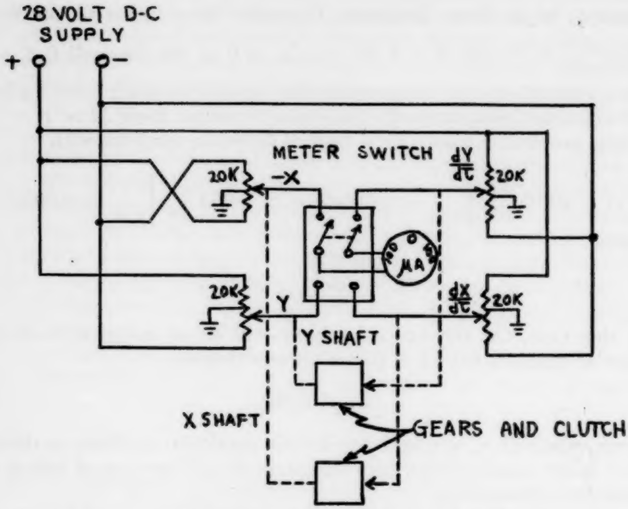


FIG. 3. Difference analyzer setup for the system

$$\frac{dX}{d\tau} = Y, \quad \frac{dY}{d\tau} = -X.$$

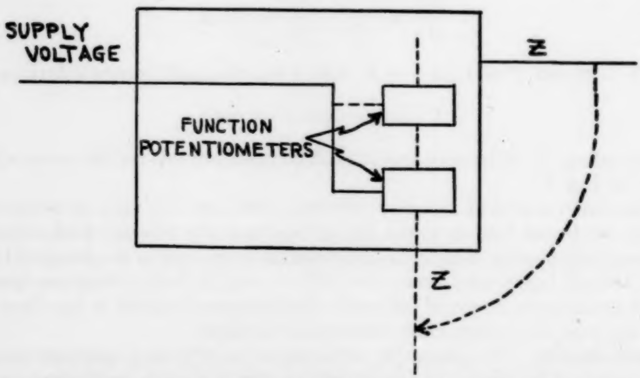


FIG. 4. Mechanical amplification and function generation. The $Z$-shaft is set to the correct value by means of a meter or comparison potentiometer (no integration).

a voltage $Z$ corresponding to some function

$$Z = Z(X, Y, \tau)$$

of the independent and dependent variables is brought out to a terminal from some point of the computing networks, read by means of a meter or comparison potentiometer, and re-entered into the machine by means of a $Z$-shaft. Potentiometers on this $Z$-shaft may then generate desired functions of the new variable $Z$.

This process will be used most often to make a new larger voltage corresponding to $X$ available in the computer or to change impedance levels in the computing networks (mechanical amplification). It should be noted carefully that such mechanical amplification and function generation does *not* constitute an integration (no derivative of $Z$ is used to reset the $Z$-shaft). All mechanical transformations of this type should be performed *before* the integrations. It may be expedient to use knobs of a different color for shafts whose settings are not derived from integrations.

The operator will then begin each step of the computation by balancing the potentiometers used for mechanical amplification and will then proceed by balancing the potentiometers used for integration in the manner outlined further above. While mechanical transformations do introduce extra shafts which must be set at each step of the computation, their use may result in substantial improvements and simplifications in the computer setup.

**The Automatic Difference Analyzer.** Whereas the difference analyzer, as described above, may be operated by relatively unskilled personnel, even the simple operations required for the stepwise computation method could be mechanized through the use of very simple servomechanisms, thereby permitting completely automatic operation of the equipment. Fig. 5 shows the block diagram of an automatic difference analyzer for the solution of the equation

$$\frac{dX}{d\tau} = f(X, \tau).$$

The computing networks function as before. The comparison potentiometer becomes the follow-up potentiometer of a servomechanism which balances the potentiometer by making the shaft displacement equal to $\frac{dX}{d\tau}$.

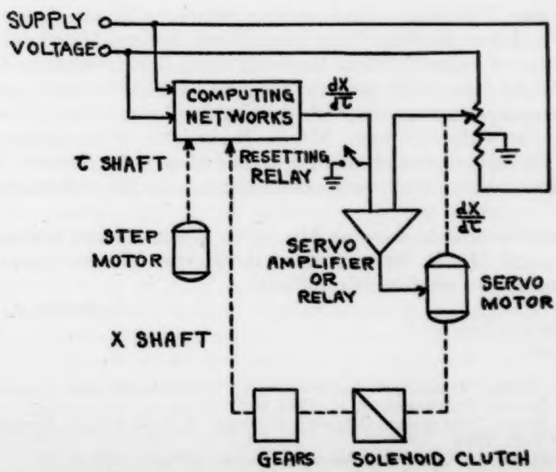


FIG. 5. Automatic difference analyzer. The solenoid clutch and the resetting relay are operated periodically by a switch on the $\tau$-shaft.

The servo shaft is then reset to zero by grounding one servo input by means of a resetting relay.

During this process, the resetting clutch shown is engaged by means of a solenoid in such a manner that the correct increment $\Delta X$ is added to the displacement of the $X$-shaft through a gear train. At the same time, the $\Delta\tau$ shaft displacement is increased by a specified amount by means of a step motor; the resetting relay as well as the solenoid clutch are actuated by a switch on this shaft.

The entire process is repeated until stopped by a limit switch or by the operator at some desired time $\tau$; each step must allow sufficient time for the potentiometer shafts to assume their new positions. It should be noted here that the frequency response of the servomechanism used will *not* affect the accuracy of the computation. The process described constitutes a completely automatic stepwise integration of the given differential equations by means of components such as simple relay servos which are cheaper than the integrators used in a comparable differential analyzer.

**Conclusions.** The difference analyzer seems to be an easily constructed, reliable, and useful instrument of about slide rule accuracy and should be well adapted to the solution of sets of complicated non-linear differential equations in smaller laboratories and educational institutions. In such cases, the cost of a differential analyzer of comparable accuracy and reliability would often be prohibitive, and the equations would yield only to tedious numerical integration. If still greater accuracy is desired, a difference analyzer solution will provide a convenient check on conventional numerical analysis. F. J. MURRAY has also suggested the use of stepwise integration methods based on non-linear extrapolation in order to decrease the number of steps needed and to increase the accuracy of the machine.

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GRANINO A. KORN

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# An Extension of Gauss' Transformation for Improving the Condition of Systems of Linear Equations

**1. Gauss' Transformation Extended.** Consider a consistent system of linear equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \; (i = 1, \cdots, n) \tag{1}$$

with $a_{ij}$, $b_i$ real. Let the matrix be symmetric and of positive rank $n - d$ and suppose the quadratic form corresponding to $A$ is non-negative semi-definite. Thus the solution points of (1) in affine $n$-space form a linear subspace of dimension $d$.

The following is our extension of a transformation due to Gauss: Let $s = (s_1, \cdots, s_n)$ be any real vector. Make the substitution

$$x_i = y_i + s_i y_{n+1} \; (i = 1, \cdots, n), \tag{2}$$

and thereby convert (1) into a system (3) of $n$ equations in the $n + 1$ unknowns $y_1, \cdots, y_{n+1}$:

$$\sum_{j=1}^{n} a_{ij}y_j + \left( \sum_{j=1}^{n} a_{ij}s_j \right) y_{n+1} = b_i \; (i = 1, \cdots, n). \tag{3}$$

An $(n + 1)$-th equation is obtained as the weighted sum of the $n$ equations (3):

$$\sum_{j=1}^{n} \left( \sum_{i=1}^{n} a_{ij}s_i \right) y_j + \left( \sum_{i,j=1}^{n} a_{ij}s_i s_j \right) y_{n+1} = \sum_{i=1}^{n} b_i s_i. \tag{4}$$

The redundancy of (4) means that the solution space of the equation pair (3, 4) is a linear subspace of dimension $d + 1$; that is, the rank of the coefficient matrix $A_1$ of the system (3, 4) is $n - d$. However, the quantities $x_i = y_i + s_i y_{n+1}$ are determined exactly as well by the system (3, 4) as by the system (1). If $A$ is symmetric, the system (3, 4) also has a symmetric coefficient matrix.

Gauss[9,10], in writing how he liked to solve certain systems (1) by relaxation,[23] presented a transformation whose application, he was convinced, would improve the convergence of the relaxation process for normal equations associated with the adjustment of surveying data. Gauss' transformation was originally presented only for non-singular ($d = 0$) systems (1), and was the special case $s_1 = \cdots = s_n = -1$ of (2). The same transformation was given by Dedekind[6], who showed its effectiveness in one example. Zurmühl[22] brings the apparently forgotten transformation to light again, but errs in asserting that it will speed the solution by relaxation and by Seidel's method of all (non-singular) systems of equations for which the respective method is slowly convergent.[24]

In two letters Gauss[9,10] reveals the motivation of his transformation $x_i = y_i - y_{n+1}$ in these terms: By the method of least squares he is seeking to determine the values of $n + 1$ quantities $y_1, \cdots, y_{n+1}$ (e.g., azimuths or elevations), whose magnitudes can be deduced from the given data up to

an additive constant. To use the equations (1) amounts to selecting the origin so that $y_{n+1} = 0$ (e.g., measuring angles from one of the unknown azimuths). But how may one decide which unknown to set equal to zero? In this quandary Gauss[10] warns us [p. 251] not to set *any* of the unknowns equal to zero, but to leave them all variable, and then to determine their *differences* by solving the system (3, 4). This, Gauss is convinced, will lead to faster convergence of the relaxation process, because of the symmetrical treatment of all the variables. Incidentally, one also gains an attractive column-sum check as a control on accuracy.

We shall examine the effect of transformation (2) on the system (1) from a different point of view. We shall ascribe a "condition number" $P(A)$ to the matrix $A$, whether singular or not. We shall show the effect on $P(A)$ of the transformation (2) and, in particular, show when $P(A)$ can be lowered and by how much. As tools we use an extension of a separation lemma known in many connections—for example, for the one-step escalator process for eigenvalues.[14] By repeated application of the extended lemma we derive a $k$-step separation theorem, believed new, applicable, for example, to the $k$-step escalator process.[2,8]

For non-singular matrices $A$, CESARI[4,3] has considered the relation between $P(A)$ and $P[\pi(A)]$, where $\pi(A)$ is a polynomial in $A$.

For positive definite matrices $A$ the relation of $P(A)$ to the accuracy of the solution of (1) by elimination is discussed at length by VON NEUMANN & GOLDSTINE.[16]

**2. Condition of a Singular Matrix.** The condition of a system $Ax = b$ with $|A| \neq 0$ describes the influence of small changes in $A$ and $b$ on $x$; the larger the change in $x$ for given changes in $A$ and $b$, the "worse" the condition. Though the condition depends also on $b$, the numbers hitherto proposed (see TODD[18]) to measure the condition are functions solely of $A$. When $A$ is not singular, Todd suggests the ratio $P = |\lambda_i|_{\max}/|\lambda_i|_{\min}$ as a condition number of $A$, where the $\lambda_i$ are the eigenvalues of $A$. In the following, however, we are concerned with systems $Ax = b$, where $A$ may be a singular matrix. Then the solutions form a linear subspace $X$, and it is the displacement of this linear subspace which should be dealt with by a condition number. Cutting with a linear subspace $V$ orthogonal and complementary to $X$, we can measure the displacement of $X$ by the displacement of its intersection $x$ with $V$. But $x$ is the unique common point of the intersections of the hyperplanes $Ax = b$ with $V$. We may therefore measure the condition of the singular system $Ax = b$ by the condition of the related non-singular problem in $V$. We are thus led to the following definition of a condition number:

*Let the eigenvalues $\lambda_i$ of $A$ be numbered so that*

$$(5) \quad 0 = \lambda_1 = \cdots = \lambda_d < |\lambda_{d+1}| \leq |\lambda_{d+2}| \leq \cdots \leq |\lambda_n| \; (0 \leq d < n).$$

*The condition number $P(A)$ of $A$ is defined as the ratio $|\lambda_n|/|\lambda_{d+1}|$ of the maximum and minimum absolute value of the non-vanishing eigenvalues.*

For non-negative, semi-definite $A$ all $\lambda_i$ are real and non-negative. For such $A$ we shall study the effect of the transformation (2) on $P(A)$.

The sensitivity of $x$ or $X$ to changes of the coefficients in (1) probably has a decisive influence on the speed of convergence of an iterative solution of (1). Eigenvalues of $A$ which are exactly zero do not seem to be troublesome in iterative methods of solving the system. In the *gradient method* (see 6, for example) all iterations take place in some subspace $V$ orthogonal to the

solution space $X$, and one gets to some point of $X$ without difficulty. For the gradient method an immediate extension of theorems of KANTOROVICH[13] proves that the $A$ length[27] of the error vector decreases per step by at least the factor $[P(A) - 1][P(A) + 1]^{-1}$. Perhaps $P(A)$ bears a direct relation to the rate of convergence of iterative processes which are invariant under rotations of the axes. Also, it might ordinarily give some indication of the convergence of processes (like *relaxation* and the methods of Seidel[17] and JACOBI[12]) which are not invariant.

**3. Eigenvalues of the Transformed Matrix.** To study the effect on $P$ of the transformation (2), we may without loss of generality choose an origin so that each $b_i = 0$ and choose axes so that $A$ is in diagonal form: $a_{ij} = \lambda_i \cdot \delta_{ij}$, where the $\lambda_i$ are numbered as in (5). Because of the semi-definiteness of $A$, this can be achieved by a real transformation. The $s_i$ are subjected to the same transformation. Then finding some solution of the $d$-fold indeterminate system (1) is equivalent to finding some point in the subspace of centers of the family of similar elliptic cylinders

$$\sum_{i=1}^{n} \lambda_i x_i^2 = \text{const.} \tag{6}$$

In the variables $y_i$ defined by (2) the quadrics (6) become a new family of elliptic cylinders. Finding some solution of the $(d + 1)$-fold indeterminate system (3, 4), and hence some solution of (1), is equivalent to finding some point in the subspace of centers of the transformed quadrics

$$Q(y_1, \cdots, y_{n+1}) \equiv \sum_{i=1}^{n} \lambda_i (y_i + s_i y_{n+1})^2 = \text{const.} \tag{7}$$

The geometrical effect of the transformation (2) is easily visualized for a non-singular $(d = 0)$ matrix $A$ in two dimensions $(n = 2)$. Solving $A$ is equivalent to finding the common center of the ellipses (6). In the variables $y_i$ defined by (2) the ellipses (6) become a family (7) of elliptic cylinders. Each cylinder of (7) is generated by elements parallel to the direction $(s_1, s_2, -1)$ passing through an ellipse (6). Getting a point $(y_1, y_2, y_3)$ on the axis of the cylinders (7) is equivalent to finding one solution of (3, 4). This one solution of (3, 4) yields the unique solution of (1). Note that the eccentricity $\epsilon = \{1 - [P(A_1)]^{-2}\}^{\frac{1}{2}}$ of the elliptical normal sections of the cylinders can be varied in the range $0 \leq \epsilon < 1$ by varying the vector $s = (s_1, s_2)$. The best "condition" of $A_1$ corresponds to a circular cross section, for which $\epsilon = 0$ and $P(A_1) = 1$.

Let $\mu_0 \leq \mu_1 \leq \cdots \leq \mu_n$ be the eigenvalues of the quadratic form (7), whose matrix $A_1$ is the coefficient matrix of the system (3, 4). The $\mu_i$ are the roots of the determinantal equation

$$\begin{vmatrix} \lambda_1 - \mu & 0 & \cdots & 0 & \lambda_1 s_1 \\ 0 & \lambda_2 - \mu & \cdots & 0 & \lambda_2 s_2 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & \lambda_n - \mu & \lambda_n s_n \\ \lambda_1 s_1 & \lambda_2 s_2 & \cdots & \lambda_n s_n & \sum_{i=1}^{n} \lambda_i s_i^2 - \mu \end{vmatrix} = 0. \tag{8}$$

Expansion of (8) according to the last row and column gives the equation

$$
\begin{aligned}
0 &= \left(\sum_{i=1}^{n} \lambda_i s_i^2 - \mu\right) \prod_{k=1}^{n} (\lambda_k - \mu) - \sum_{i=1}^{n} \lambda_i^2 s_i^2 \prod_{\substack{k=1 \\ (k \neq i)}}^{n} (\lambda_k - \mu) \\
&= -\mu\left[\prod_{k=1}^{n} (\lambda_k - \mu) + \sum_{i=1}^{n} \lambda_i s_i^2 \prod_{\substack{k=1 \\ (k \neq i)}}^{n} (\lambda_k - \mu)\right].
\end{aligned} \tag{9}
$$

The factor $\mu^{d+1}$ can be removed from (9), since $\lambda_1 = \cdots = \lambda_d = 0$. There remains the following equation for the other $\mu_i$:

$$
\prod_{k=d+1}^{n} (\lambda_k - \mu) + \sum_{i=d+1}^{n} \lambda_i s_i^2 \prod_{\substack{k=d+1 \\ (k \neq i)}}^{n} (\lambda_k - \mu) = 0. \tag{10}
$$

We now state the principal tool in the study of (2):

LEMMA. I. *For any real numbers $s_i$ and any set of $\lambda_i$ satisfying* (5), *the roots $\mu_i$ of* (8) *have the following properties:* (*i*) *exactly $d + 1$ of the $\mu_i$ are zero;* (*ii*) *the remaining $n - d$ roots $\mu_i$ satisfy the following separation condition:*

$$
0 < \lambda_{d+1} \leq \mu_{d+1} \leq \lambda_{d+2} \leq \mu_{d+2} \leq \cdots \leq \lambda_n \leq \mu_n < \infty. \tag{11}
$$

II. *Conversely, given any $\mu_{d+1}, \cdots, \mu_n$ satisfying* (11), *one can determine real numbers $s_i$ so that the roots of* (8) *are* $0, 0, \cdots, 0, \mu_{d+1}, \cdots, \mu_n$.

PROOF. *Of* I. Case 1. No $s_i = 0$; $\lambda_{d+1} < \lambda_{d+2} < \cdots < \lambda_n$. We can divide (10) through by $\prod(\lambda_k - \mu)$, getting the equation

$$
f(\mu) \equiv \sum_{i=d+1}^{n} \frac{\lambda_i s_i^2}{\mu - \lambda_i} - 1 = 0. \tag{12}
$$

Since (12) shows that $f(0) < -1$ and since we previously removed a factor $\mu^{d+1}$, we have proved (*i*). Since each $\lambda_i s_i^2 > 0$, a sketch of $f(\mu)$ shows at once that

$$
\lambda_{d+1} < \mu_{d+1} < \lambda_{d+2} < \mu_{d+2} < \cdots < \lambda_n < \mu_n, \tag{13}
$$

proving (*ii*).

Case 2. $s_i$, $\lambda_i$ unrestricted. Since the roots $\mu_i$ of (8) are continuous[25] functions of the $\lambda_i$ and the $s_i$, (11) follows from (13) by a passage to the limit.

*Of* II. Since the choice of $s_1, \cdots, s_d$ is arbitrary, we have only to determine real $s_{d+1}, \cdots, s_n$. This is equivalent to determining non-negative $\lambda_i s_i^2$ $(d + 1 \leq i \leq n)$ so that the roots $\mu$ of (10) are the given $\mu_{d+1}, \cdots, \mu_n$.

Case 1. (13) holds. Then equations (10) and (12) are equivalent. But the roots of (12) are the ellipsoidal (confocal) coordinates corresponding to the cartesian coordinates $\{\sqrt{\lambda_i s_i^2}\}$ in $(n - d)$-space. The following inversion formulas give the $\{\lambda_i s_i^2\}$ as rational functions of the $\mu_j$ and $\lambda_j$; [see[21], p. 548]:

$$
\lambda_i s_i^2 = \prod_{j=d+1}^{n} (\mu_j - \lambda_i) \Big/ \prod_{\substack{j=d+1 \\ (j \neq i)}}^{n} (\lambda_j - \lambda_i) > 0 \ (i = d + 1, \cdots, n). \tag{14}
$$

Hence $s_{d+1}, \cdots, s_n$ are uniquely determined as positive functions of the $\lambda_j$ and $\mu_j$, where $\mu_j$ are the roots of (10).

Case 2. $\lambda_j$, $\mu_j$ are restricted only by (11). We shall replace all $\lambda_j$, $\mu_j$ by neighboring values $\lambda_j'$, $\mu_j'$ which satisfy (13) and also the following condition: for each $j$ where $\lambda_j = \mu_j = \lambda_{j+1}$, we insist that

$$\mu_j' = \tfrac{1}{2}(\lambda_j' + \lambda_{j+1}'). \tag{15}$$

By Case 1, let real $s_i'(d + 1 \leq i \leq n)$ be determined so that (14) and (10) hold for the primed symbols. Now let $\lambda_j' \to \lambda_j$, $\mu_j' \to \mu_j$. By (15), the $\lambda_i' s_i'^2$ of (14) all approach (non-negative) limits, which we define to be $\lambda_i s_i^2$. Since the left-hand side of (10) is a continuous function of the arguments $\lambda_j$, $\mu_j$, $s_j^2$, it is seen that (10) is satisfied in the limit. In this manner we have proved the existence of real $s_{d+1}, \cdots, s_n$ in Case 2. (The $s_{d+1}, \cdots, s_n$ need not be unique in Case 2.)

Equation (12) is a special case of the one-step escalator equation of MORRIS.[14] Similar equations occur in the generalized RAYLEIGH-RITZ method of ARONSZAJN[1] (which includes the MORRIS escalator as a special case), in dealing with the realizability of impedance functions by electrical networks[11], and in defining ellipsoidal coordinates.[21] In all these connections Part I of the lemma is known for the case of unequal $\lambda_i$.

The lemma may readily be extended to diagonal matrices $A$ with arbitrary real $\lambda_i$, although we will not use it. Conclusion ($i$) continues to hold. In addition to condition (11) for the positive $\lambda_i$, $\mu_i$, there is a similar condition for the negative $\lambda_i$, $\mu_i$, in which, for each $i$, $\lambda_{i+1} \leq \mu_i \leq \lambda_i < 0$.

**4. Effect on the Condition Number.** Our condition number for the matrix $A$ is $P(A) = \lambda_n/\lambda_{d+1}$, while the same for the matrix $A_1$ is $P(A_1) = \mu_n/\mu_{d+1}$. The dependence of $P(A_1)$ on the $\lambda_i$ and the $s_i$ can ordinarily be stated only in terms of the roots of (8), but certain general remarks can be made:

(a) The lemma shows that $P(A_1)$ can always be made greater than $P(A)$ by some choice of $s$, and that, unless $\lambda_{d+1} = \lambda_{d+2}$, $P(A_1)$ can also be made less than $P(A)$.

(b) A most favorable choice of $s$ is one for which $\mu_{d+1} = \lambda_{d+2}$ and $\mu_n = \lambda_n$, so that $P(A_1) = \lambda_n/\lambda_{d+2}$. This can be brought about by making (for this particular coordinate system) $s_{d+1} \neq 0$, $s_i = 0$ $(i \neq d + 1)$, whence the roots of (8) are 0 ($d + 1$ times), $\lambda_{d+2}, \lambda_{d+3}, \cdots, \lambda_n$, and $(s_{d+1}^2 + 1)\lambda_{d+1}$. Then $\mu_{d+1} = \lambda_{d+2}$, $\mu_n = \lambda_n$ if and only if $\lambda_{d+2} \leq (s_{d+1}^2 + 1)\lambda_{d+1} \leq \lambda_n$, or

$$\frac{\lambda_{d+2} - \lambda_{d+1}}{\lambda_{d+1}} \leq s_{d+1}^2 \leq \frac{\lambda_n - \lambda_{d+1}}{\lambda_{d+1}}. \tag{16}$$

In particular, we can choose

$$s_{d+1}^2 = (\lambda_n - \lambda_{d+1})/\lambda_{d+1} = P(A) - 1. \tag{17}$$

(c) For a matrix $A$ not in diagonal form the selection of $s$ such that $s_{d+1}$ satisfies (17) and such that the other $s_i = 0$ can be made as soon as we know $\lambda_{d+1}$, $\lambda_n$, and the eigenvector $u_{d+1}$ belonging to $\lambda_{d+1}$. At least in the usual case $d = 0$ the $\lambda_{d+1}$, $\lambda_n$, $u_{d+1}$ can ordinarily be approximated by known procedures. To know the least value of $P(A_1)$ achievable by the transformation (2) requires knowledge of $\lambda_{d+2}$ also. Conversely, if $s$ has been selected so that $\mu_{d+1} = \lambda_{d+2}$, the determination of $\mu_{d+1}$, the least non-zero eigenvalue of $A_1$,

yields $\lambda_{d+2}$. Regarded as a matrix transformation to assist in the determination of the higher eigenvalues of $A$, this resembles a transformation of Tucker.[19]

(d) If $\lambda_{d+1}$, $\lambda_n$, $u_{d+1}$ are known only roughly, we can expect to make $P(A_1)$ reasonably close to its minimum $\lambda_n/\lambda_{d+2}$ by picking $s$ in the direction of the rough value of $u_{d+1}$, with $|s|^2$ equal to the rough value of $(\lambda_n - \lambda_{d+1})/\lambda_{d+1}$.

**5. Repeated Application of the Transformation. General Separation Theorem.** The transformation (2) can be applied a second time, to generate a matrix $A_2$ of rank $n - d$ in the $n + 2$ variables $z_1, \cdots, z_{n+2}$. This time 0 becomes a $(d + 2)$-fold multiple eigenvalue of $A_2$, and the separation formula (11) relates the eigenvalues of $A_1$ to those of $A_2$. Finally, the variables $z_i$ and $x_i$ are related by the formula

$$z_i = x_i + s_i z_{n+1} + t_i z_{n+2}. \tag{18}$$

The substitution (18) would border $A$ in one step with two new rows and two new columns. Clearly it is possible for $P(A_2)$ to get as low as $\lambda_n/\lambda_{d+3}$.

If the generalized Gauss transformation (2) is applied $k$ times, we get a matrix $A_k$ of order $n + k$ and rank $n - d$. We have the following theorem, proved by $k$ applications of the lemma:

Theorem. *The $n + k$ eigenvalues $\kappa_{-k+1}, \cdots, \kappa_0, \kappa_1, \cdots, \kappa_n$ of the matrix $A_k$ have the following properties:* (i) *exactly $d + k$ of the $\kappa_i$ are zero;* (ii) *the remaining $n - d$ values $\kappa_i$ can be numbered so as to satisfy the following inequalities:*

$$\begin{cases} \kappa_{d+1} \le \kappa_{d+2} \le \cdots \le \kappa_n; \\ \lambda_i \le \kappa_i \le \lambda_{i+k} \ (d + 1 \le i \le n - k); \\ \lambda_i \le \kappa_i < \infty \ (n - k < i \le n). \end{cases} \tag{19}$$

*Conversely, given any $\kappa_{d+1}, \cdots, \kappa_n$ satisfying the inequalities* (19), *one can determine $k$ real transformations* (2) *so that $A_k$, the $k$-th successive transform of $A$, has eigenvalues* $0, 0, \cdots, 0, \kappa_{d+1}, \cdots, \kappa_n$.

After $k$ generalized Gaussian transformations (2), we see that $P(A_k)$ can theoretically be made as low as $\lambda_n/\lambda_{d+k}$. After $n - d$ transformations $P(A_{n-d})$ can be made equal to $\lambda_n/\lambda_n = 1$; at this stage the equations are perfectly conditioned.

The theorem can be extended to diagonal matrices $A$ with arbitrary real $\lambda_i$. In the extension one gets inequalities of the type

$$\lambda'_{i+k} \le \kappa_i' \le \lambda_i'$$

corresponding to negative eigenvalues

$$\lambda'_{i+k} \le \cdots \le \lambda'_{i+1} \le \lambda_i' \le \cdots < 0$$

of $A$. In the extended form the theorem is applicable to the $k$-step $(k > 1)$ escalator process of Aronszajn[22] for symmetric matrices $A$, described by Fox.[8]

**6. Example.** For a certain class **C** of matrices $A$ it is known *a priori* that $d = 0$ and that $u_1$ has components which are all positive and roughly equal. **C** includes matrices like (21), which correspond to the Dirichlet problem over a finite net or to related random walk problems. **C** also includes the matrices of the normal equations for the angle variables or the altitudes

in a survey; this was the source of the examples in[9,10,5]. If $A$ belongs to **C**, the original form [$s = (-1, \cdots, -1)$ in unnormalized coordinates] of Gauss' transformation is close to the optimal selection of $s$ parallel to $u_1$.

On the other hand, if $A$ is such that $d = 0$ but $(-1, \cdots, -1)$ is, roughly speaking, closer to $u_n$ than to $u_1$, Gauss' form of (2) is likely to make $P(A_1) > P(A)$, whereas a choice of $s$ near $u_1$ will make $P(A_1) \leq P(A)$.

As an example of this, we cite T. S. WILSON's ill-conditioned matrix

$$A = \begin{bmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{bmatrix} \tag{20}$$

given by Todd.[18] The decomposition of $A$ is as follows:[26]

$$\begin{aligned}
\lambda_1 &= 0.01015, & u_1 &= (-.830, \ .501, \ .208, -.124); \\
\lambda_2 &= 0.8431\ , & u_2 &= (\ .094, -.302, \ .761, -.568); \\
\lambda_3 &= 3.858\ , & u_3 &= (\ .396, \ .614, -.271, -.625); \\
\lambda_4 &= 30.29\ , & u_4 &= (\ .380, \ .526, \ .552, \ .521).
\end{aligned}$$

Hence $P(A) = 2984$, a relatively high value.

Gauss' original transformation, corresponding to $s = (-1, -1, -1, -1)$ in unnormalized coordinates, would replace $A$ by the coefficient matrix

$$A_1 = \begin{bmatrix} 5 & 7 & 6 & 5 & -23 \\ 7 & 10 & 8 & 7 & -32 \\ 6 & 8 & 10 & 9 & -33 \\ 5 & 7 & 9 & 10 & -31 \\ -23 & -32 & -33 & -31 & 119 \end{bmatrix}.$$

To obtain the condition number of $A_1$, we find the components $s_i$ of $s$ in the coordinate system of the eigenvectors $u_i$:

$$s_1 = .245, \ s_2 = .015, \ s_3 = -.114, \ s_4 = -1.979.$$

With these $s_i$, equation (12) becomes

$$\frac{.00061}{\mu - .01015} + \frac{.0002}{\mu - .8431} + \frac{.050}{\mu - 3.858} + \frac{118.6}{\mu - 30.29} = 1.$$

Hence the eigenvalues of $A_1$ are approximately

$$\mu_1 = .01027, \ \mu_2 = .8431, \ \mu_3 = 3.867, \ \mu_4 = 148.9,$$

so that $P(A_1) \doteq 14500$. As measured by $P$, the matrix $A_1$ resulting from Gauss' original transformation is even worse conditioned than $A$.

On the other hand, some rough knowledge of $\lambda_1, \lambda_4$ permits considerable reduction in $P$. Following the principles of section 4 but using only one figure of $u_1$, we select $s$ in the direction $(.8, -.5, -.2, .1)$. To satisfy (17) approximately we multiply this vector by 50, getting $s = (40, -25, -10, 5)$. With these weights we obtain the transformed matrix

$$A_1' = \begin{bmatrix} 5 & 7 & 6 & 5 & -10 \\ 7 & 10 & 8 & 7 & -15 \\ 6 & 8 & 10 & 9 & -15 \\ 5 & 7 & 9 & 10 & -15 \\ -10 & -15 & -15 & -15 & 50 \end{bmatrix}.$$

The components of $s$ in the normalized coordinate system are

$$s_1 = -48.42,\ s_2 = .86,\ s_3 = .075,\ s_4 = -.865.$$

With these $s_i$ equation (12) becomes

$$\frac{23.797}{\mu - .01015} + \frac{.623}{\mu - .8431} + \frac{.0217}{\mu - 3.858} + \frac{22.66}{\mu - 30.29} = 1.$$

Hence the eigenvalues of $A_1'$ are approximately

$$\mu_1 = .8205,\ \mu_2 = 3.853,\ \mu_3 = 11.21,\ \mu_4 = 66.22,$$

so that $P(A_1') \doteq 80.71$. Thus $A_1'$ is far better conditioned than $A$. If we had selected $s$ *exactly* parallel to $u_1$, $P(A_1')$ could have been made as low as $(30.29)/(.8431) = 35.93$.

**7. Pairing of the Eigenvalues.** By the lemma it is theoretically always possible for $n$ even, $d = 0$, to make the non-zero eigenvalues $\mu_i$ occur in pairs, even though all $\lambda_i$ are distinct; cf.[20]. If so, in solving (3, 4) by the gradient method, for example, the double roots $\mu_i$ of $A_1$ act like single roots and the essential dimensionality of the calculation is reduced from $n$ to $n/2$. For example, consider

$$(21)\quad A = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \quad (2m \text{ rows}),$$

whose eigenvalues are $4 \sin^2 [k\pi/(2m+1)]$ $(k = 1, 2, \cdots, 2m)$. If $s = (-1, \cdots, -1)$, the transformation (2) yields the matrix

$$(22)\quad A_1 = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \quad (2m+1 \text{ rows}),$$

whose eigenvalues are $4 \sin^2 [2k\pi/(2m+1)]$ $(k = 0, 1, 1, 2, 2, \cdots, m, m)$.

These two matrices are related to those for discrete random walks in one dimension: (21) to walks on a line segment with "manholes" at both ends, and (22) to walks on the circumference of a circle, with no manholes. The matrices are also used in solving the Dirichlet problem on a discrete net.

NBSINA
Univ. of Calif.
Los Angeles

George E. Forsythe
Theodore S. Motzkin

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[21] E. T. Whittaker & G. N. Watson, *A Course of Modern Analysis*. American edition, New York, 1943, p. 547.

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[23] Gauss was very fond of relaxation, which is identical with the process summarized by Fox.[7] Gauss[9] remarked that the process was so easy that he could do it while half asleep or while thinking about other things.

[24] Consider the system $x + ry = 0$, $rx + y = 0$, $|r| < 1$. When $r = \pm (1 - \epsilon)$, either method converges slowly. Gauss' transformation with $s = (-1, -1)$ very much speeds the relaxation solution and the Seidel solution for $r = -(1 - \epsilon)$; but for $r = +(1 - \epsilon)$ it does not affect the relaxation solution and worsens the Seidel solution.

[25] For a simple proof of this well known fact, see Motzkin.[15]

[26] The calculations of this section were performed by Mrs. Louise Straus.

[27] The $A$ length of a vector $u$ is $(\Sigma a_{ij}u_iu_j)^{\frac{1}{2}}$.

# RECENT MATHEMATICAL TABLES

**933[A, K].**—M. T. L. Bizley, "A note on the variance-ratio distribution," Institute of Actuaries Students' Soc., *Jn.*, v. 10, 1950, p. 62–64.

A folded page following p. 64 gives a table of the binomial coefficients $\binom{m}{t}$ for $t \leq m = 1(1)20$. The table is used in this case to facilitate the evaluation of

$$\int_u^\infty y^n(y + k)^{-m-1}dy.$$

D. H. L.

**934[A, D].**—L.Îa Neĭshuler, *Tablitsy Perevoda Prîamougol'nykh Dekartovykh Koordinat v Polîarnye* [Tables for the Transformation of Rectangular Cartesian into Polar Coordinates]. Moscow and Leningrad, Gostekhizdat, 1950, 292 p. 16 × 33.6 cm. Boards, 33.75 roubles.

This is the fourth, and by far the most elaborate, table of Neĭshuler to which we have made reference (see *MTAC*, v. 1, p. 7, for tables of 1930 and 1933; v. 2, p. 203–204, for table of 1945). It was published in November 1950, in an edition of 5000 copies, and seems to have met a real need, since the edition was sold out in less than a year.

It is a table giving the polar coordinates $(s, \alpha)$ corresponding to rectangular coordinates $(x, y)$, where $s = (x^2 + y^2)^{\frac{1}{2}}$, $\alpha = \arctan(y/x)$; $x$ and $y$ are positive integers up to 10000. The main table occupies very large quarto pages 7–290. For using the tables it is generally supposed that $y$ is not less than $x$. If in a given case $x$ is the greater, the $x$ is called $y^*$ and the angle $(90° - \alpha)$.

The first column on each page is for $y$, and the first range of values is 1000 to 1090. To this range, beginning on page 7, and ending with the first column on page 12, corresponds a series of ranges for $x$: 0–10(5), 10–20(15), $\cdots$, 180–200(190), 520–550(535), $\cdots$, 1070–1105(1105), the maximum terminal value of $x$ approximating to 1090—the maximum terminal value of $y$. The next range of $y$ is 1090–1180, and the last 9910–10000.

Under each of the $x$-ranges are columns headed $s$, $\alpha$, $\Delta$. The values in the $s$ and $\alpha$ columns correspond exactly to the series of $x$ values above in brackets ( ), namely: 5, 15, $\cdots$, 190, $\cdots$, 535, $\cdots$, 1105. All of the values of $x$ in this series except the last (a range boundary-value) are means of the end-values of the $x$-ranges. Throughout the table the means are almost invariably chosen. The $s$ and $\alpha$ corresponding to other values of $x$, are got by interpolation—using the values of $\Delta s$ and $\Delta\alpha$ given in the $\Delta$ column. In the range of $y$ chosen as illustration the $\Delta s$ and $\Delta\alpha$ never have more than 9 entries but on the last pages of the table these run up to 22.

For values of $y$ less than 1000, such for example, as $y = 12$, $x = 9$, the table-entry for $y = 1200$, $x = 900$ would naturally be used. Similarly for other cases of $y$ less than 1000, the final results obtained for $s$ are usually to one place of decimals, and the angle to the nearest tenth of a minute. Illustrative examples are worked out on pages 4–6. No previously published table of this kind compares with it in extent.

The small tables of W. J. Seeley, and of J. C. P. Miller have been re-

viewed in *MTAC*, v. 2, p. 22–23, and the single-page table of Jahnke & Emde, p. 17 of the more elementary section, is well known.

From an entry in *MTAC*, v. 3, p. 340, we learned that among "Tables completed or almost completed," by 1948, were "Cartesian to Polar Conversion Tables," supervised by E. H. Neville: "To give, for integral values of $x, y$, with $y \leq x \leq 105$, values to 12 figures of $r$ with $\theta$ in degrees, and of ln $r$ with $\theta$ in radians."

R. C. Archibald

Brown University
Providence, R. I.

935[A].—H. S. Uhler, (a) "Many-figure approximations to $\sqrt{2}$, and distribution of digits in $\sqrt{2}$ and $1/\sqrt{2}$," Nat. Acad. Sci., Washington, *Proc.*, v. 37, 1951, p. 63–67. (b) "Approximations exceeding 1300 decimals for $\sqrt{3}$, $1/\sqrt{3}$, sin $(\pi/3)$ and distribution of digits in them," Nat. Acad. Sci., Washington, *Proc.*, v. 37, 1951, p. 443–447.

In (a) the value of $\sqrt{2}$ is given to 1544D. In (b) values of $\sqrt{3}$, $1/\sqrt{3}$, and sin $(\pi/3) = \frac{1}{2}\sqrt{3}$ are given to 1316D. In (a) are given data on the distribution of the digits in $\sqrt{2}$ and $1/\sqrt{2}$, namely, the values of $\chi^2$ and the corresponding probabilities of obtaining such distributions from a normal population [*MTAC*, v. 4, p. 109]. These data are remarkably small for $1/\sqrt{2}$ but reasonably sized for $\sqrt{2}$. Corresponding data are given for the constants in (b). These are all fairly large. A detailed comparison of distributions of digits in $1/\sqrt{2}$ and $1/\sqrt{3}$ is given in a separate table.

Agreement with Coustal's value of $\sqrt{2}$ [*MTAC*, v. 4, p. 144] was exact.

D. H. L.

936[D, F, L]—John Todd. *Table of Arctangents of Rational Numbers.* National Bureau of Standards AMS No. 11, Washington 1951, xii + 105 p. 26 × 19.7 cm. Price $1.50.

This work gives two tables. Table 1 gives for every pair of integers $m$, $n$ with $0 < m < n \leq 100$ the principal values of arctan $m/n$ and arccot $m/n$ in radians to 12D. In addition are given $m^2 + n^2$ and the canonical representation of arctan $m/n$ as a linear combination, with integer coefficients, of "irreducible" arctangents of integers.

For $r$ an integer, arctan $r$ is called reducible in case it can be expressed as a linear combination of arctangents of integers less than $r$. Table 2 gives a list of reducible arctangents of integers $\leq$ 2089 together with the unique reductions in terms of irreducible arctangents. Thus the entry

$$601 \quad 2(1) + 1(24) - 1(25)$$

is a statement of the fact that

$$\arctan 601 = 2 \arctan 1 + \arctan 24 - \arctan 25.$$

For further properties of these reducible arctangents see *MTAC* v. 2, p. 62–63, 147–148, v. 4, p. 82–83, 85.

The table was produced by punched card methods. Besides its theoretical interest the table is very useful in computing the logarithms of complex numbers belonging to a rectangular grid.

D. H. L.

**937[F, G].**—H. Gupta. "A generalization of the partition function," Indian Acad. Sci., *Proc.*, v. 17, 1951, p. 231–238.

The author denotes by $v_r(n, m)$ the number of partitions of $n$ into parts not exceeding $m$, each part $k$ being of $k^{r-1}$ different types.

The generating function of $v_r(n, m)$ is

$$\prod_{k=1}^{m} (1 - x^k)^{-k^{r-1}} = \sum_{n=0}^{\infty} v_r(n, m)x^n.$$

For $r = 1$, $v_r(n, m)$ becomes the familiar function first tabulated by Euler. The author gives a table of $v_2(n, m)$ for $1 \leq m \leq n \leq 50$.

The function may be built up from the recursive relation

$$nv_r(n, m) = \sum_{k=1}^{n} \sigma_r(k, m)v_r(n - k, m)$$

in which $\sigma_r(k, m)$ denotes the sum of the $r$-th powers of those divisors of $k$ which do not exceed $m$.

The author states that

$$v_2(n, m) = \exp \{n^{\frac{2}{3}}(C + o(1))\},$$

where $C^3 = 27\zeta(3)/4$ so that $C = 2.009$.

However, $50^{-\frac{2}{3}} \ln v_2(50,50) = 1.700$.

D. H. L.

**938[F].**—A. Katz. "Third list of factorization of Fibonacci numbers," *Riveon Lemat.* v. 5, 1951, p. 13.

New factors of $U_n$ or $V_n$ are given as follows:

| $n$ | $U_n$ or $V_n$ | Factor |
|---|---|---|
| 138 | V | 16561, 1043766587 |
| 141 | U | 108289 |
| 147 | U | 3528 |
| 147 | V | 65269 |
| 153 | V | 13159 |
| 165 | U | 86461 |
| 180 | V | 8641 |
| 189 | U | 38933 |
| 189 | V | 85429 |

The residual factors are in every case greater than 200000.

D. H. L.

**939[F].**—Ernst S. Selmer "The Diophantine equation $ax^3 + by^3 + cz^3 = 0$." *Acta Math.*, v. 85, 1951, p. 203–362.

The extensive tables at the close of this paper should be very useful in the further study of cubic diophantine equations. The main table on page 348 lists all equations $x^3 + my^3 + nz^3 = 0$ for $2 \leqq m < n < 50$, stating in nearly every case whether the equation is soluble or insoluble in integers, and if soluble giving an absolutely least integral solution. A second group of tables on pages 349–353 lists the soluble and insoluble equations $ax^3 + by^3 + cz^3 = 0$ with $abc$ cube-free and $\leqq 500$ and $a$, $b$, $c$ positive co-prime integers. Finally

on page 357 there is a table listing the number of generators and the basic solutions of the equation $X^3 + Y^3 = AZ^3$ with $A$ cube-free and $\leqq 500$. Apparently the only previously published table is one given by FADDEEV[1] for this last equation extending up to $A = 50$.

There are in addition some useful auxiliary tables of cubic residues in pure cubic fields $K(\sqrt[3]{m})$, $m < 50$ and the EISENSTEIN field $K(P)$. In table 3 on page 351, the author has noted that the entry for $w$ when $\lambda = 0$ and $p = 17$ should be $w = 1$ instead of $w = 0$.

MORGAN WARD

California Institute of Technology
Pasadena, Calif.

[1] D. K. FADDEEV, "Ob uravnenii [on the equation] $x^3 + y^3 = Az^3$," Akad. Nauk S.S.S.R., Leningrad, Fiz-mat. Inst. imeni V. A. Stekloff, *Trudy*, v. 5, 1934, p. 25–40.

**940[H].**—A. ZAVROTSKY. "Tablas para la resolucion de las ecuaciones de quinto grado," Acad. de Cien. Fis., Mat. Nat., Caracas, *Boletin*, v. 13, 1950, p. 51–93.

The author reduces the general quintic equation to the form

$$x^5 = px^3 + qx^2 + rx + 1.$$

The tables give the least positive or greatest positive root of this equation according as $p + q + r$ is negative or positive. The coefficients $p$, $q$, $r$, range over all integers not exceeding 10 in absolute value. Values of the roots are given to 5D. Each entry of the table was computed separately by one of a number of iterative methods. For a table on the cubic by the same author, see *MTAC*, v. 2, p. 28–30.

D. H. L.

**941[K].**—W. E. DEMING, *Some Theory of Sampling*. New York, John Wiley & Sons, 1950. xvii + 602 p., 15.9 × 23.5 cm. $9.00.

Table 1, page 558, Fiducial Factors between $s$ and $\sigma$, seems to be new.

The distribution of the standard deviation $s$ in random samples of size $n$ from a normal population of standard deviation $\sigma$ is given by

$$f(s)ds = n^{\frac{1}{2}(n-1)}2^{\frac{1}{2}(5-n)}(n-1)^{-1}s^{n-2}\sigma^{1-n}\exp\{-\tfrac{1}{2}ns^2\sigma^{-2}\}ds \tag{1}$$

Writing

$$P_s = \int_s^\infty f(t)dt \tag{2}$$

it is evident that (2) is an incomplete $\Gamma$-function in which $s$ and $\sigma$ occur only in the form $s/\sigma$ (let $s/\sigma = t$ in (1)). For each value of $P_s$ and $n$ equation (2) is satisfied by a value of $s/\sigma$ which is $1/f_{100P_s}$ in Deming's notation. Letting $ns^2/\sigma^2 = u$ in (1), it is evident that $u$ is distributed as $\chi^2$ with $n - 1$ degrees of freedom. Thus

$$P_s = P(\chi^2) = .95, .50,$$

where

$$\chi^2 = ns^2\sigma^{-2} = n/f_{P_s}^2. \tag{3}$$

Given $n$ and $P_s$, a value of $\chi^2$ is found satisfying

$$P(\chi^2) = \int_{\chi^2}^{\infty} f(\chi^2)d\chi^2$$

for $n - 1$ degrees of freedom, and $f_{P_s}$ is immediately given by (3). Tables of $f_{95}$ and $f_{50}$ are useful in finding the .05 and .50 fiducial limits of $\sigma$, given $s$ and $n$. Table 1 gives values of $f_{50}$ and $f_{95}$ to 6D for $n = 2(1)25$.

H. A. FREEMAN

Massachusetts Institute of Technology
Cambridge, Mass.

**942[K].**—W. J. DIXON, "Ratios involving extreme values," *Annals of Math. Stat.*, v. 22, 1951, p. 68–78.

Let $x_1, \cdots, x_n$ represent the values of a sample of size $n$ from a normal population arranged in increasing order of magnitude. Let

$$\begin{aligned} r_{10} &= (x_n - x_{n-1})/(x_n - x_1), r_{11} = (x_n - x_{n-1})/(x_n - x_2), \\ r_{12} &= (x_n - x_{n-1})/(x_n - x_3), r_{20} = (x_n - x_{n-2})/(x_n - x_1), \\ r_{21} &= (x_n - x_{n-2})/(x_n - x_2), r_{22} = (x_n - x_{n-2})/(x_n - x_3). \end{aligned}$$

The six tables of this paper present the values of $R$ which satisfy the relation

$$Pr(r_{ij} > R) = \alpha,$$

for $i = 1, 2$; $j = 0, 1, 2$; $n = 2 + i + j$ (1) 30; $\alpha = .005, .01, .02, .05, .10$ (.10) .90, .95.

J. E. WALSH

Bureau of the Census
Washington, D. C.

**943[K].**—E. J. GUMBEL & J. A. GREENWOOD, "Table of the asymptotic distribution of the second extreme," *Annals Math. Stat.*, v. 22, 1951, p. 121–124.

Consideration is given to the asymptotic distribution of the next to last and of the second value of a large sample from an initial distribution of the exponential type. Table I enables one to test (using the asymptotic distribution) the hypothesis that a sample came from a completely prescribed population by means of the second largest (penultimate) variate. A transformation is given which enables one to use the table for the second smallest value. Probability values are given to 5D (with a method to obtain more places if desired) with second central differences for $y_2 = -1.95$ (.05) 5.25, 5.35, 5.50, 5.65, 5.90, 6.45, where $y_2 = \frac{1}{2}n(x_2 - u_2) f(u_2)$, in which $n$ is the sample size, $f(x) = F'(x)$ is the initial density function, and $u_2$ is defined by $F(u_2) = \dfrac{n-2}{n}$. Probability points for $y_2$ to 5D are given in Table II for .005, .01, .025, .05, .1, .25, .5, .75, .9, .95, .975, .990, .995. Mention is made of applying the tabular values in the construction of a probability paper, but insufficient information is presented to enable one actually to follow the proposal without using the references.

C. F. KOSSACK

Purdue University
Lafayette, Ind.

**944[K].**—F. J. MASSEY, JR., "The distribution of the maximum deviation between two sample cumulative step functions," *Annals Math. Stat.*, v. 22, 1951, p. 125–128.

Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_m$ be the ordered results of two random samples from populations having continuous cumulative distribution functions $F(x)$ and $G(x)$, respectively. Let $S_n(x) = k/n$ where $k$ is the number of observed values of $X$ which are less than or equal to $x$, and similarly let $S_m'(y) = j/m$ where $j$ is the number of observed values of $Y$ which are less than or equal to $y$. The statistic $d = \max_x |S_n(x) - S_m'(x)|$ can be used to test the hypothesis $F(x) \equiv G(x)$, where the hypothesis is to be rejected if the observed $d$ is significantly large. The limiting distribution of $d(mn/(m+n))^{\frac{1}{2}}$ has been derived[1] and tabulated.[2] In this paper the author describes a method for obtaining the exact distribution of $d$ for small samples and a short table for equal size samples is included. In Table 1, the probability of $d \leqq k/n$ is given for $n = m$, $k = 1(1)12$, $n = 1(1)40$ usually to 6D.

L. A. AROIAN

Hughes Aircraft Company
Culver City, Calif.

[1] N. SMIRNOV, "On the estimation of the discrepancy between empirical curves of distribution for two independent samples," Moscow, Univ., *Bull. Math.*, (*série internationale*), v. 2, 1939, fasc. 2, 16p.

[2] N. SMIRNOV, "Table for estimating the goodness of fit of empirical distributions," *Annals Math. Stat.*, v. 19, 1948, p. 279–281.

**945[K].**—F. J. MASSEY, JR., "The Kolmogorov-Smirnov test for goodness of fit," Am. Stat. Assn., *Jn.*, v. 46, 1951, p. 68–78.

Let $F_0(x)$ be a continuous population cumulative distribution, $S_N(x)$ the observed cumulative step-function of a sample (i.e., $S_N(x) = k/N$, where $k$ is the number of observations less than or equal to $x$), then the sampling distribution of $d$ = maximum $|F_0(x) - S_N(x)|$ is known, and is independent of $F_0(x)$. In Table 1, the author gives the prob$\{\max|S_N(x) - F_0(x)| > d_\alpha(N)\} = \alpha$, for $\alpha = .20, .15, .10, .05$, and $.01$, $N = 1(1)20$ to 3D; and for $N = 25, 30$, and 35 to 2D. For $N > 35$ the limiting values of SMIRNOV[1] apply. In Table 3 the author compares the minimum deviation of actual from assumed population that is detectable with probability .50 by the $\chi^2$ and $d$ tests at the 5 percent and 1 percent levels of significance, with $N = 200(50)1000(100)1500(500)2000$, for values of $\chi^2$ to 4D and values of $d$ to 3D. (The $\chi^2$ portion of this table is from WILLIAMS.[2])

L. A. AROIAN

Hughes Aircraft Company
Culver City, Calif.

[1] N. SMIRNOV, "Table for estimating the goodness of fit of empirical distributions," *Annals Math. Stat.*, v. 19, 1948, p. 279–281.

[2] C. A. WILLIAMS JR., "On the choice of the number and width of classes for the Chi-square test for goodness of fit," Amer. Stat. Assn., *Jn.*, v. 45, 1950, p. 77–86.

**946[K].**—H. S. SICHEL, "The estimation of the parameters of a negative binomial distribution with special reference to psychological data," *Psychometrika*, v. 16, 1951, p. 107–127.

The negative binomial distribution is written in the form,

$$f(r) = \frac{\Gamma(r+p)}{\Gamma(p)\,\Gamma(r+1)} \left(\frac{p}{p+m}\right)^p \left(\frac{m}{p+m}\right)^r, \tag{1}$$

in which optimum estimates of $p$ and $m$ are uncorrelated. The moment estimate $\bar{m}$ = mean, is shown to be efficient and the moment estimate, $\bar{p}$ = (mean)$^2$/(variance–mean), to have efficiency given by

$$\text{Eff. }(\bar{p}) = \left[ 2 \sum_2^\infty \frac{1}{r} \left(\frac{m}{p+m}\right)^{r-2} \frac{\Gamma(r)\,\Gamma(p+2)}{\Gamma(r+p)} \right]^{-1}. \tag{2}$$

These results are essentially those given by Fisher.[1] Figure 1 gives Eff. $(\bar{p})$ for $p$ varying continuously from 0 to 3 and $m = .1, .5, 1(1)4, \infty$. It is suggested (as also by Fisher) that Eff. $(\bar{p})$ may be fairly accurately estimated by using $\bar{m}$ and $\bar{p}$ in (2) above.

The maximum likelihood estimate, $\hat{p}$, is the solution of

$$\frac{1}{n} \sum_1^n [\psi(\hat{p} + r_i - 1) - \psi(\hat{p} - 1)] - \log\left(1 + \frac{\hat{m}}{\hat{p}}\right) = 0,$$

where $\hat{m} = \bar{m} = \frac{1}{n} \sum r_i$. This equation is given by HALDANE[2] as his equation (2.1). The remainder of the paper, except for three examples of applications, is given over to tables useful in solving this equation. Table 1 gives values of $\lambda(r,\hat{p}) = \psi(\hat{p} + r - 1) - \psi(\hat{p} - 1)$ to 5D for $r = 0(1)35$ and $\hat{p} = .1(.1)3.0$. Table 2 gives values of $\psi(\hat{p} - 1)$ to 5D for $\hat{p} = .1(.1)3.0$. The author seems to be unaware of the existence of the numerous earlier tables of the digamma function.[3] Checks with the earlier tables indicate some errors in Table 1; Table 2 is correct.

LEO KATZ

Michigan State College
East Lansing, Mich.

[1] R. A. FISHER, "The negative binomial distribution," *Annals of Eugenics*, v. 11, 1941, p. 182–187.
[2] J. B. S. HALDANE, "The fitting of binomial distributions," *Annals of Eugenics*, v. 11, 1941, p. 179–181.
[3] See FMR *Index*, p. 202–203.

**947[L].**—M. ABRAMOWITZ, "Tables of the functions $\int_0^\phi \sin^{1/3} x\,dx$ and $(4/3) \sin^{-4/3} \phi \int_0^\phi \sin^{1/3} x\,dx$," N.B.S. *Jn. of Research*, v. 47, 1951, p. 288–290.

The first integral is given to 8D for $\phi = 0°(1°)90°$, the second integral to 8D for $\phi = 0°(30')90°$ and to 7D for $\phi = 90°(30')180°$.

A. E.

**948[L].**—B. P. BOGERT, "Some roots of an equation involving Bessel functions," *Jn. Math. Phys.*, v. 30, 1951, p. 102–105.

Table I gives 4 to 6 S values of the first root $x$ of

$$J_0(x)Y_1(kx) - Y_0(x)J_1(kx) = 0 \tag{1}$$

together with 4S (in one case 5S) values of

$$y = \frac{2}{\pi}(k-1)x$$

for $k$ = 1,1.1,1.2,1.25,1.3, 1.4,1.5,1.59334, 1.6(.1) 2.1,2.45882, 3(1)10, 20.

Table II gives 4 to 5 S values of the first root of equation (1) for $k$ = 1 (.01)2, 2.1, 3(1)10, 20.

Table III gives 4 S values of the first root of

$$Y_1(x)J_0(kx) - J_1(x)Y_0(kx) = 0 \tag{2}$$

for $.05 \leqslant k \leqslant 1$ (irregular interval, 111 values)

In addition to entries listed in *MTAC*, v. 1, p. 222, the author refers to the following related tables:

CARSTEN & MCKERROW—unpublished manuscripts, equation (1) for $k = (1.1)^{1/2}$, 1.06(.02)1.2(.5)1.5,2(1)5; 2–3D.

RENDULIC, *Wasserwirtschaft und Technik*, no. 25–26, 1935, p. 270, equation (1), $k$ = 0(.1)1(1)7, 6.25; 3–4 D.

DURANT, equation (2), see *MTAC*, v. 2, p. 172.

A. E.

**949[L].**—COMPUTATION DEPARTMENT OF THE MATHEMATICAL CENTER, Amsterdam. Interim reports nos. R 53, Int. 1–8. The oscillating wing in a subsonic flow. R 53, Int. 1. (1949), 7 p., 5 Datasheets. R 53, Int. 2. (1949), 21 p. R 53, Int. 3. (1950), 2 p., 9 Datasheets (reproduced on 17 p.). R 53, Int. 4. (1950), 12 p. R 53, Int. 5. (1950), 12 p. R 53, Int. 6. (1951), 2 p., 42 Datasheets. R 53, Int. 7. (1951), 35 p. + 3 p. of corrections. R 52, Int. 8. (1951), 2 p., 18 Datasheets. 21.5 × 32.6 cm.

The Computation Department of the Amsterdam Mathematical Center has carried out voluminous computations on behalf of the (Dutch) National Aeronautical Research Institute. The final report will not be available for general distribution, but a certain amount of unclassified information regarding expansions and methods of computation of certain functions, and a number of working sheets, are released in a series of interim reports. Seven of the eight interim reports carry the same title; R 53, Int. 5. is entitled "Expansions of $B_m^{(n)}$ and $a_n$ into power-series with respect to $\tau$." The first report carries a loose sheet with a list of the 23 members of the Computing Section of the Computation Department. The effort to make accessible to the general computing public as much information as is consistent with security, is praiseworthy and perhaps worthy of imitation.

The numerous expansions and descriptions of computations contained in this report will prove very valuable to future workers in this field. It is impossible to describe them concisely beyond saying that the larger part refers to Mathieu functions, and the lesser part to certain integrals related to Bessel functions. The work on Mathieu functions is a useful supplement to MCLACHLAN's book[1] whose notations the authors follow.

A description of the Datasheets (mostly to 8 D or 8 S) follows.

R 53, Int. 1. Gegevenblad 1. General orientation about parameters. Gegevenblad 2. For $\tau = .1(.05).35$ this gives the corresponding values of $\cos \eta_1 = 1 - 2\tau$ (on the worksheet $\sin \eta_1$ appears, presumably a mistake or a survival from an earlier notation), and then $\sin n\, \eta_1$ for $n = 1(1)27$. Datasheets 3, 3a. Values of $J_n(\beta^2\Omega)$ for $n = 0(1)26$ and $\beta^2\Omega$ from .0175 to 11.2 at varying intervals (59 values). Spotchecking a few values against the Harvard University tables of Bessel functions[2] revealed no discrepancy. Datasheet 4 gives $n^{-1} \sin n\, \eta_1$ for the same values of $n$ and $\eta_1$ as in Datasheet 2. Actually, the table is given twice: first $n$ increases, and then $n$ decreases, in the columns. This is to facilitate computation of convolution sums like

$$\sum_{n=2}^{m-1} \frac{\sin n\, \eta_1}{n} \frac{\sin (m-n)\, \eta_1}{m-n}.$$

R 53, Int. 3 gives numerical values of the Fourier coefficients and characteristic values of odd Mathieu functions. Mathieu's equation is written in the form

$$y'' + (a - 2q \cos 2x) = 0, \quad q = k^2$$

and the function

$$se_n(x,q) = \sum_{m=1}^{\infty} B_m^{(n)}(q) \sin mx$$

is that solution of Mathieu's equation which has period $2\pi$, reduces to $\sin nx$ when $q = 0$, and for which

$$\sum_{m=1}^{\infty} [B_m^{(n)}(q)]^2 = 1.$$

The corresponding characteristic values of $a$ are $b_n(q)$. Datasheets 5 to 12 (each comes in two parts) give $B_m^{(n)}(q)$ for $k = .025(.025).15(.05).3$ $(.075).45(.1).65$, .8,1,1.2,1.5,2,3 and $n = 1(1)8$. The range for $m$ varies, $m = 33$ being the highest value that occurs (for $n = 1$ and two values of $k$). Thus, quite apart from the question of different normalization, there are values in these worksheets which are not covered by the NBSCL Mathieu function tables.[3] Datasheet 13 gives $b_n$ for the same values of $k$ and $n$ as in the previous datasheets, and in addition for $n = 9(1)12$ for a few values of $k$. Since $k$ rather than $q$ has been taken as a variable, some of the values do not occur in the NBSCL tables (which tabulate $b_n + 2k^2$ as a function of $4k^2$) although they all fall within the range covered there. Spotchecking a few values of $b_n$ with the NBSCL tables did not reveal any discrepancy beyond one unit of the last decimal place (of the Amsterdam table) which is well within the limits of accuracy the authors claim.

R 53, Int. 6. Datasheets 14 to 55 give values of

$$\int_0^{\eta_1} \cos (n\eta) \exp (i\beta^2\, \Omega \cos \eta) d\eta$$

for $n = 0(1)20$, $\tau = \frac{1}{2} - \frac{1}{2} \cos \eta_1 = .1(.05).35$ and for varying ranges of values of $\beta^2\Omega$ between 0 and 4.8. Real and imaginary parts are given on separate sheets.

R 53, Int. 8. Datasheets 60 to 71 gives values of

$$\int_0^x J_0(t) \begin{matrix}\cos\\ \sin\end{matrix}\, ct\, dt, \int_0^x Y_0(t) \begin{matrix}\cos\\ \sin\end{matrix}\, ct\, dt$$

for $c = .3(.1).8$ and $x = 0(.1)6.1$. The report states that in the heading of the tables $\frac{x}{c}$ should be replaced by $cx$. These tables exceed both in their range and, partly, in the number of decimals the tables by Schwarz[4] who introduced these functions. Datasheets 72–77 give the real and imaginary parts (both parts on the same sheet) of

$$\int_0^\infty (c + \cosh \xi)^{-1} \exp(-ix \cosh \xi) d\xi$$

for $c = .3(.1).8$ and $x = 0(.1)6.1$.

A. E.

[1] N. W. McLachlan, *Theory and Application of Mathieu Functions*. Oxford University Press, 1947.

[2] Harvard University Computation Laboratory, *Annals* v. 3–14, *Tables of the Bessel Functions of the First Kind of Orders Zero through Hundred Thirty-Five*. Harvard University Press, 1947–1951.

[3] NBSCL, *Tables Relating to Mathieu Functions*, New York, Columbia University Press, 1951.

[4] See *MTAC*, v. 1, 1944, p. 248, 250, 304.

950[L].—Wolfgang Gröbner & Nikolaus Hofreiter, *Integraltafel. Zweiter Teil: Bestimmte Integrale.* Vienna and Innsbruck, Springer-Verlag, 1950, vi, 204 p. 20.7 × 29.8 cm.

The first volume of this work contains indefinite integrals, and it was reviewed in *MTAC*, v. 3, 1949, p. 482. The present second volume lists principally those definite integrals which cannot be evaluated in a simple manner from the indefinite integrals of the first volume. In addition, many integrals which could be computed from the first volume have been included for the convenience of the user. For instance, the indefinite integral

$$\int x^{2m}(x^2 + a^2)^{-n-\frac{1}{2}}dx, \quad n > m \geq 0$$

is given in volume I as a (finite) series. If the limits 0 and $\infty$ are substituted in that series, one obtains a sum containing binomial coefficients. In volume II the integral (from 0 to $\infty$) is given in closed form, thus saving the user the labor of summing the series.

Known definite integrals are very numerous, and in order to prevent the book from becoming unwieldly the authors limited themselves to a selection. By introducing variable parameters they often give master formulas from which many integrals can be computed, and do not list too many of the particular cases. All integrals recorded in the table have been carefully checked, and to make assurance doubly sure, each entry is accompanied by a coded instruction telling the user how to verify the result if he wishes to do so.

The general arrangement of the second volume closely resembles that of the first volume, except that there are five sections in volume II instead of the three sections of volume I.

The Introduction contains a list of symbols and notations, a list of references, methods for the evaluation of definite integrals, and general formulas. In the list of references Dirichlet's lectures on definite integrals, Bierens de Haan's tables of integrals, Magnus & Oberhettinger's book

on special functions, WATSON's *Bessel Functions*, and WHITTAKER & WATSON's *Modern Analysis* are mentioned, but not the volume of corrections by LINDMAN to Bierens de Haan's tables, nor the integral tables by RYZHIK (*MTAC*, v. 1, 1945, p. 442–443).

Section 1, Rational integrands (21 p.) contains also formulas for, and integrals involving, the classical orthogonal polynomials. Section 2, Algebraic integrands (20 p.) contains also elliptic integrals both in Legendre's normal form and in Weierstrass' canonical form. Section 3, Elementary transcendental integrands (117 p.) contains many integrals which can be evaluated in terms of error functions, Bessel functions, and other higher transcendental functions. There are also special subsections devoted to Euler's dilogarithm and to the exponential integral and related functions. In this section there is a table of Laplace transforms (with a reference to DOETSCH's book of 1937, but without reference to any of the more recent and more extensive tables of Laplace integrals). No table of Fourier transforms is given, but many Fourier integrals occur in various subsections of section 3. The last subsection in this section lists information on limits of definite integrals: Dirichlet's singular integral, the integral occurring in the Riemann-Lebesgue lemma, and Laplace transforms.

The last two sections contain material which has no counterpart in the first volume. Section 4 Euler integrals (18 p.) contains some hypergeometric integrals in addition to integrals which can be evaluated in terms of the gamma function and related functions. Section 5, Bessel functions (20 p.) contains both integrals which can be evaluated in terms of Bessel functions and integrals whose integrands contain Bessel functions.

The book is an offset print from an excellent handwritten copy. In comparison with the first volume, one finds larger letters and explanations written in italic rather than script characters, both features contributing to legibility.

A. E.

**951[L].**—Y. L. LUKE & M. A. DENGLER, "Tables of the Theodorsen circulation function for generalized motion," *Jn. Aeron. Sci.*, v. 18, 1951, p. 478–483.

The function in question is

$$C(z) = \frac{H_1^{(2)}(z)}{H_1^{(2)}(z) + iH_0^{(2)}(z)} = F(\rho, \theta) + iG(\rho, \theta),$$

where $z = \rho e^{i\theta}$, and $H_n^{(2)}$ is the Hankel function. Tables 1 and 2 give 7D values of $F$ and of $-G$ for $\theta = -5°(5°)30°$ and $\rho = 0(.01).3(.02).5(.05)1(.5)4(1)10$. Table 3 gives 7D values of $F$ and of $-G$ for $\theta = 0$ and $\rho = k = 0(.002).1(.01).3(.02).34(.01).36(.02).44(.01).46(.02).54(.01).56(.02).64(.01).66(.02).74(.01).76(.02).84(.01).86(.02).94(.01).96(.02)1(.1)2(.5)10(10)50,100,\infty$. All tables were computed from standard American and British tables of Bessel functions, and the error is estimated by the authors to be at most 3 units of the seventh decimal.

A. E.

**952[L].**—NBSCL, *Tables Relating to Mathieu Functions*. Columbia University Press, New York, 1951. xlvii + 278 p. 19.7 × 26.7 cm. Price $8.00.

These tables now make it possible to calculate radiation and scattering from slits, strips and elliptic cylinders with nearly the same facility as has been previously possible for rods and spheres.

Solutions of the Helmholtz equation $\nabla^2\psi + k^2\psi = 0$ may be required in solving the wave equation, the diffusion equation, and, in the limiting case of zero potential, the Schroedinger equation. Factored solutions are possible for the eleven coordinate systems for which the equation separates. For most of these systems one or two of the coordinates are "angle" coordinates, with a finite range of values; the rest are "radial" coordinates, with a range from 0 to $\infty$ or from $-\infty$ to $\infty$. To solve most problems of physical interest one needs tables of the eigenfunction solutions for the angle coordinates, with corresponding eigenvalues for the separation constant; one also needs *both* independent solutions for the radial coordinates, for the eigenvalues of the separation constant.

Heretofore adequate tables have been available for only three coordinate systems: rectangular, for which the solutions are trigonometric and hyperbolic functions; circular cylindrical for which the Bessel and Neumann functions are needed;[1] and the spherical, for which tables of spherical harmonics are also required.[2] Other systems of practical interest are the elliptic cylinder coordinates, with solutions tabulated in the tables here reviewed, the parabolic cylinder, the parabolic and the spheroidal coordinates, for which adequate tables are still lacking.

Factors for elliptic cylinder coordinates are solutions of the separated equation

$$y'' + (b + s\cos^2 x)\,y = 0,$$

where the "angle" coordinate corresponds to real values of $x$ from 0 to $2\pi$ and the "radial" coordinate to imaginary values, from 0 to $i\infty$. The tables under review provide the foundation for radiation and scattering calculations for $s$ (which is proportional to the square of the ratio between the interfocal distance to the wave-length) from 0 to 100, with enough entries to allow reasonably accurate second-difference interpolation.

Eigenvalues of the separation constant $b$ are given for the first 31 periodic eigenfunctions for the angle coordinate, to 8 decimal places. Corresponding values of the coefficients of the Fourier series expansions of these eigenfunctions are given to allow a 9D accuracy of the resulting Fourier series. These eigenfunctions alternate in symmetry, of course; the first being even about $x = 0$, the next being odd and so on. The series coefficients are normalized so that the value of the even functions, $Se_m$, are unity and the slopes of the odd functions, $So_m$, are unity at $x = 0$. This normalization has some advantage in radiation calculations, though the alternative normalization, that the quadratic integral over $x$ from 0 to $2\pi$ be $2\pi$, is preferred by some. In the former case the values of the solution for $x$ near $\pi/2$, for $s$ large, may be quite large; in the second case the corresponding values for $x$ near 0 may be vanishingly small. Since the ratio between the coefficients for the two normalizations is tabulated, either one may be computed.

In the case of the radial functions the preference is more clear-cut. Most

radiation or scattering problems involve the asymptotic amplitudes of the two independent solutions. Because of the simple relationship between the Fourier series for the eigenfunction and the Bessel-function series for the radial solutions, the former normalization (value or slope unity) results in a simple asymptotic form for the solutions, the latter (integral of the square = $2\pi$) does not. From the coefficients, as tabulated, these simpler forms of the two radial solutions may be computed as functions of $-ix$ by using tables of Bessel and Neumann functions. Values and slopes of the two radial functions at $x = 0$ are given to 8 significant figures in the volume, thus allowing computation of radiation and scattering from degenerate elliptic cylinders (strip or slit) to be carried out without the use of additional Bessel-function tables. Again interpolation is made easy by the type of presentation.

For those with easy access to large computing machines the Tables in this volume will probably allow easy calculation of most radiation or scattering or resonance problems of practical interest. For those having only desk equipment available it would save much labor to have subsidiary tables of the actual angle and radial functions published, though these additional tables would be quite bulky because of the large number of entries required (different values of $s$, of $m$ and of $x$ all requiring tabulation).

The Mathematical Tables Project is again to be congratulated in carrying out, in so satisfactory a manner, the immense amount of labor required to obtain the entries in this volume. One can hope that the spheriodal functions will eventually be tabulated with similar accuracy and extensiveness.

PHILIP M. MORSE

Mass. Inst. of Tech.
Cambridge, Mass.

[1] See, for example: *Scattering and Radiation from Circular Cylinders and Spheres; Tables of Amplitudes and Phase Angles* [*MTAC*, v. 3, p. 107]. For other tables of Bessel Functions see *MTAC*, v. 1, p. 205–308. The new tables of Bessel Functions [*MTAC*, v. 5, p. 223–224] published by the Harvard Computation Laboratory will be most useful as soon as the corresponding second solutions are published.

[2] See, for example: H. TALLQVIST, "Tafel der 24 ersten Kugelfunktionen $P_n(\cos\theta)$ [*MTAC*, v. 1, p. 4], and also MTP, *Tables of the Associated Legendre Functions* [*MTAC*, v. 1, p. 164–165] and *Tables of Spherical Bessel Functions* [*MTAC*, v. 3, p. 26].

**953[L]**.—MARTHA PETSCHACHER, "Tabelle di Funzioni Ipergeometriche," Rome, Univ., Ist. Naz., Alta Mat., *Rend. Mat. e Appl.*, s. 5, v. 9, 1951, p. 389–420.

For the calculation by the hodograph method of steady motions of a gas in two dimensions, two families of hypergeometric functions are of importance. The first, which is relevant for finding the Legendre potential and position co-ordinates of the gas-flow, is

$$F(\tau) = F(\alpha_\mu, \beta_\mu, \mu + 1; \tau), \tag{1}$$

where $F$ is the hypergeometric series and $\tau$ is the variable, $\mu$ a parameter, and $\alpha_\mu$, $\beta_\mu$ are determined from

$$\alpha_\mu + \beta_\mu = \mu + 1/(\gamma - 1), \quad \alpha_\mu\beta_\mu = -\tfrac{1}{2}\mu(\mu - 1)/(\gamma - 1), \tag{2}$$

$\gamma$ being the adiabatic index of the gas. The second, which is relevant for finding the stream function, is

$$F(a_\mu, b_\mu; \mu + 1; \tau), \tag{3}$$

where

$$a_\mu + b_\mu = \mu - 1/(\gamma - 1), \quad a_\mu b_\mu = -\tfrac{1}{2}\mu(\mu + 1)/(\gamma - 1).$$

For physical significance the range of $\tau$ is $0 \leq \tau < 1$.

In the present publication the function $F(\tau)$ defined by (1), (2) is tabulated for the case $y = 1.4$, for $\tau = 0(.02)1$ and $\mu = \frac{1}{8}(\frac{1}{8})\frac{7}{8}; \frac{3}{4}(\frac{1}{4})2(\frac{1}{2})7(1)10$, to 6D throughout. A user of these tables is, however, more likely to be interested in significant figures than decimals; the number of significant figures is 6 or 7 for the smaller values of $\mu$, diminishing to 4 (or, for a small part of the table, 3) for the higher $\mu$. The related function

$$Y(\tau) = \tau^{\mu/2} F(\alpha_\mu, \beta_\mu; \mu + 1; \tau)/\Gamma(1 + \mu)$$

also is given; the number of significant figures is in general the same as for $F(\tau)$.

The calculation was made (a) for $0 \leq \tau \leq 0.7$ by numerical solution of the hypergeometric differential equation, (b) for $0.7 \leq \tau < 1$ by expressing (1) in terms of hypergeometric functions of $1 - \tau$ and finding these from their power series; the agreement at $\tau = 0.7$ provided an over-all check. The table is stated to be correct to about half a unit in the last figure, and the reviewer has verified this for a few entries.

The function (3), and its derivatives, have been tabulated by FERGUSON & LIGHTHILL[1] and by VERA HUCKEL,[2] and from these the function (1) can be easily found. But the present table breaks new ground in covering the range $\frac{1}{2} \leq \tau < 1$ and including values of $\mu$ which are neither integral nor half-integral. However, for the gas-flow context one requires also the derivative of $F(\tau)$, and it is to be hoped that the author will extract this from her work sheets and publish it.

T. M. CHERRY

University of Melbourne
Melbourne, Australia

[1] D. F. FERGUSON & M. J. LIGHTHILL, "The hodograph transformation in transonic flow," Roy. Soc. London, *Proc.*, v. 192A, 1947, p. 135–142.

[2] V. HUCKEL, *Tables of Hypergeometric Functions for Use in Compressible-Flow Theory*. N. A. C. A. *Technical Note* no. 1716, Washington, 1948.

954[L].—G. PÓLYA & G. SZEGÖ. *Isoperimetric inequalities in mathematical physics*. Annals of Mathematics Studies no. 27. Princeton University Press, 1951. xvi + 279 p. 17.7 × 25.3 cm. Price $3.00.

The tables (p. 247–274) give values for the following ten quantities associated with a plane domain $D$:

$L$ length of perimeter.

$A$ area.

$I$ moment of inertia of the area with respect to its centroid.

$B$. Let $a$ be any point of $D$, and $h$ the distance, from $a$, of the tangent at any point of the boundary of $D$. $B$ is the minimum, as $a$ varies over $D$, of the integral $\int h^{-1} ds$ taken over the boundary of $D$.

$\rho$ and $R$ radii of the inscribed and circumscribed circles of $D$.

$\dot{r}$ and $\bar{r}$ maximum inner radius, and outer radius, i.e., the radii of the circles (in the case of the interior, maximal circle) which bound a conformal

map of $D$, or its complement, if the linear magnification at the centre (for $D$) or at infinity (for the complement of $D$) is unity.

$P$ torsional rigidity of a cylinder of cross-section $D$.

$\Lambda$ principal frequency of a membrane of shape $D$.

The first 13 tables (p. 251–258) list the values of these constants for the following domains: circle, ellipse, square, rectangle, semi-circle, sector, 3 triangular shapes (equilateral, half of an equilateral, isosceles right-angled triangles), and regular hexagon, together with approximations for narrow ellipses, rectangles, and sectors. Not all tables list all the values. The following 14 tables (p. 259–271) list certain dimensionless combinations (for instance $\dot{r}A^{-1/2}$, $P\Lambda^2A^{-1}$) for some or all of the same domains, together, in some cases, with information about the boundedness or extreme values of the combinations in question. The last two tables (p. 272, 273) list values of $B$, $\dot{r}$, $\bar{r}$ for some additional domains.

In the main body of the monograph there are some further numerical values, inequalities, and approximations for these and similar quantities, and some corresponding quantities for three-dimensional domains. There is also (on page 22) a 4D numerical table of

$$\frac{(1 + \sin^2\delta)^{\frac{1}{4}}}{\cos(\delta/2)} \text{ and } \frac{j\Gamma[(\pi - \delta)/(2\pi)]}{2\pi^{\frac{1}{2}}\Gamma[(2\pi - \delta)/(2\pi)]}$$

for $\delta° = 0(15)60(5)85(1)90$. Here $j = 2.4048 \ldots$ is the first positive zero of the Bessel function $J_0(x)$. This table is reprinted from a paper by one of the authors.[1]

A. E.

[1] G. Pólya, "Torsional rigidity, principal frequency, electrostatic capacity and symmetrization," *Quart. Appl. Math.*, v. 6, 1948, p. 267–277.

**955[L].**—G. W. Reitwiesner. *A Table of the Factorial Numbers and their Reciprocals from* 1! *through* 1000! *to* 20 *Significant Digits*. Ballistic Research Laboratories, Technical Note no. 381. Aberdeen Proving Ground, Md., 1951. Hectographed 20.3 × 26.7 cm.

This table, whose description is quite adequately given in its title is a by-product of the computation of $e$ by the ENIAC [*MTAC*, v. 4, p. 11–15]. A similar table to 62S is on file in the Computing Laboratory of the BRL [*MTAC*, v. 5, p. 195]. This table is an extension of a previous table, Technical Note no. 106, by Lotkin [*MTAC*, v. 4, p. 15] extending to 200! with 20S. A table of $n!$ for $n = 1(1)1000$ to 16S is described in RMT **956** and UMT 69 [*MTAC*, v. 3, p. 205]. A manuscript table, identical with the one under review, by S. Johnson, is mentioned in *MTAC*, v. 3, p. 340.

**956[L].**—Lelia Ricci, "Tavola di radici di basso modulo di un'equazione interessante la scienza delle costruzioni," *Rivista di Ingegneria*, 1951, no. 2, 8 p.

Numerical tables of the first ten pairs of roots (in case of complex roots, of real and imaginary parts of the roots) of

$$\sin z = \pm kz$$

for $k = .1(.01)1$. Five decimals (claimed reliable) are given for $k = .1(.1)1$, four decimals (of which the last is not reliable) for the other values of $k$. Table I gives the complex roots, and Table II the real roots. [See also *MTAC*, v. 5, p. 231.]

A. E.

957[L].—H. E. SALZER. *Tables of $n!$ and $\Gamma(n + \frac{1}{2})$ for the First Thousand Values of $n$.* National Bureau of Standards, AMS 16. Washington 1951. Price 15 cents.

This table brings to publication a manuscript table of $n!$ already reported in *MTAC*, v. 3, p. 205. The values of $n!$ and $\Gamma(n + \frac{1}{2})$ for $n = 0(1)1000$ are given to 16S and 8S, respectively. The manuscript table of $n!$ was checked by comparison with a 24S table prepared by J. BLUM on punched card equipment.

D. H. L.

958[V].—BALLISTIC RESEARCH LABORATORIES, Report no. 757: M. LOTKIN, *Supersonic Flow of Air Around Corners*, May 1951, 20 p., 1 table, 1 diagram.

If $\mu$ be the Mach angle and the function: $f(\mu) = k^{-1}\cot^{-1}[k^{-1} \tan \mu]$ be introduced, where $k^2 = (\gamma - 1)/(\gamma + 1)$ and $\gamma$ is the specific heat ratio, then the relation between the deflection, $\delta$, of a supersonic stream and the initial and final Mach angles, $\mu_i$ and $\mu_f$, may be expressed:

$$\delta = f(\mu_f) - f(\mu_i) + \mu_f - \mu_i.$$

Also, the radial and tangential velocity components are:

$$u = c \sin [k f(\mu)]$$
$$v = c\, k \cos [k f(\mu)],$$

where $c$ is the velocity of efflux into a vacuum. The temperature, density, and pressure follow directly from the fact that $T$ is proportional to

$$\left(M^2 + \frac{2}{\gamma - 1}\right)^{-1}$$

and the adiabatic law.

Since $\delta$ occurs as the first difference of $F(\mu) = f(\mu) + \mu$, the expansion around a corner $\delta$ may be regarded as the continuation of an expansion from $M = 1$ and $\delta$ tabulated as a function of $\mu$ from $M = 1 (\mu = 90°, \delta = 0)$ to $M \to \infty \left(\mu = 0, \delta = \frac{\pi}{2}[k^{-1} - 1]\right)$. The author tabulates $\theta \equiv f(\mu)$, $\mu$, $M$, $q/c$ (where $q$ is the speed), $T/T_0$, $p/p_0$, and $\rho/\rho_0$ (where the subscript 0 refers to stagnation conditions) for 0°.5 interval of the argument $\delta$. The value $\gamma = 1.405$ is used, angles are given to 0°.001, and the thermodynamic variables are carried to five significant figures.

The author presents an unnecessarily involved form of the above synopsis of the problem, presumably motivating the highly misleading form of the only diagram in the paper which, by its contextual implication, illustrates

the physical picture of the general flow around a corner but actually applies only to the case $M = 1$.

RICHARD N. THOMAS

Univ. of Utah
Salt Lake City, Utah

**959[V].**—I. IMAI & H. HASIMOTO, "Application of the W. K. B. method to the flow of a compressible fluid, II," *Jn. Math. Phys.*, v. 28, 1950, p. 205–214.

Table 1 (p. 21) gives 4 or 5S values of $Q$ and of $Q/qK^{\frac{1}{2}}$ for $q = 0(.01)1$, where

$$K = (1 - q^2)(1 - a^2q^2)^{-1/a^2}, \quad a^2 = \frac{\gamma - 1}{\gamma + 1}, \quad \gamma = 1.4,$$

$$\mu = \left(\frac{1 - q^2}{1 - a^2q^2}\right)^{\frac{1}{2}}, \quad Q = \frac{2}{1 + \mu}\left(\frac{1 + a\mu}{1 + a}\right)^{1/a} q(1 - a^2q^2)^{(1-a)/2a}.$$

There are also some tables relating to the physical problem in hand.

A. E.

**960[Z].**—PAUL S. DWYER, *Linear Computations*. London, Chapman and Hall, New York, John Wiley and Sons, 1951, xii + 344 p. 229 × 14.4 cm. $6.50.

The main purpose of the book is to explain how to get numerical solutions for sets of simultaneous linear equations. Related matters, such as the numerical evaluation of determinants, numerical inversion of matrices, etc., are also treated, but with less detail. Attention is centered on the use of desk calculators.

Quite literally, the reader needs only to know grade school arithmetic and some high school algebra to read the first half of the book. The first three chapters discuss elementary computational matters, such as round off errors, significant digits, etc. Then there are five chapters which deal with a great many variations of the basic scheme of solving simultaneous linear equations by successive elimination of variables. Next come two chapters on determinants. The basic theorems are stated without proof, and the main attention is devoted to methods of numerical computation. A chapter on linear forms and three chapters on matrices follow, with the same pattern. Except for a thorough chapter on the errors that can arise in the above methods, the few remaining chapters seem very sketchy. For example, two related methods are given for finding the characteristic equation of a matrix, but no mention is made of well-known methods for finding the greatest eigenvalue directly. Indeed, the author seems distrustful of all iterative methods, since he dismisses even the best known and most widely used iterative methods with a bare mention.

The author is very liberal with diagrams and numerical illustrations. The explanations are full and painstaking. The book will be a great help to those with little mathematical maturity who must nonetheless personally get numerical solutions of simultaneous linear equations.

J. BARKLEY ROSSER

Cornell University
Ithaca, New York

## MATHEMATICAL TABLES—ERRATA

In this issue references have been made to Errata in RMT **939** (SELMER), **946** (SICHEL), and in Note **129**.

**198.**—R. T. BIRGE, "Least squares' fitting of data by means of polynomials," *Revs. Mod. Phys.*, v. 19, 1947, p. 298–347.

P. 341, Table XII, $n = 8$, *for* $S_{16} = 16.3635416^*$ *read* $32.727083^*$.

P. G. GUEST

Univ. of Sydney
Sydney, Australia

**199.**—R. A. FISHER & F. YATES, *Statistical Tables for Biological, Agricultural and Medical Research*. 3rd ed. New York, 1948.

These tables include five significance levels for the distribution of the variance ratio, $F$, and of $z = \frac{1}{2} \ln F$, as follows: $p = 0.2$ (calculated by H. W. NORTON), $p = 0.1$ "based on the tables of the incomplete Beta function of CATHERINE M. THOMPSON,[1] for which we are indebted to Professor E. S. PEARSON and Dr. V. G. PANSE," $p = 0.05$ and $0.01$ (calculated by R. A. FISHER), and $p = 0.001$ (attributed to C. G. COLCORD & L. S. DEMING[2]). The last is seriously infested with error, some being carried into the table of the $t$ distribution, and there are a few errors in the others, as follows:

| Page | $p$ | $n_1$ | $n_2$ | Function | *For* | *Read* |
|---|---|---|---|---|---|---|
| 34–35 | 0.2 | 4 | 1 | $z$ | 1.3097 | 1.3067 |
| | | | | $F$ | 13.73 | 13.64 |
| | | 8 | 1 | $z$ | 1.3400 | 1.3397 |
| | | | | $F$ | 14.59 | 14.58 |
| | | 24 | 2 | $z$ | 0.7452 | 0.7453 |
| | | 24˙ | 21 | $z$ | 0.1829 | 0.1831 |
| 36 | 0.1 | ∞ | 120 | $z$ | 0.0081 | 0.0881 |
| 41 | 0.01 | 1 | 2 | $F$ | 98.49 | 98.50 |
| | | 8 | 2 | $F$ | 99.36 | 99.37 |
| 42 | 0.001 | 1 | 3 | $z$ | 2.5604 | 2.5591 |
| 43 | | | | $F$ | 167.5 | 167.0 |
| 32 | | | | $t$ | 12.941 | 12.924 |
| 42 | | 1 | 5 | $z$ | 1.9255 | 1.9270 |
| 43 | | | | $F$ | 47.04 | 47.18 |
| 32 | | | | $t$ | 6.859 | 6.869 |
| 42 | | 1 | 7 | $z$ | 1.6874 | 1.6879 |
| 43 | | | | $F$ | 29.22 | 29.25 |
| 32 | | | | $t$ | 5.405 | 5.408 |
| 42 | | 1 | 40 | $z$ | 1.2674 | 1.2672 |
| | | 1 | 120 | $z$ | 1.2158 | 1.2159 |
| | | 2 | 5 | $z$ | 1.8002 | 1.8071 |
| 43 | | | | $F$ | 36.61 | 37.12 |
| 42 | | 2 | 29 | $z$ | 1.0903 | 1.0901 |
| | | 2 | 60 | $z$ | 1.0248 | 1.0250 |
| | | 2 | 120 | $z$ | 0.9952 | 0.9954 |
| 43 | | | | $F$ | 7.31 | 7.32 |

| Page | $p$ | $n_1$ | $n_2$ | Function | *For* | *Read* |
|---|---|---|---|---|---|---|
| 42 | 0.001 | 3 | 1 | $z$ | 6.5966 | 6.6000 |
| | (cont.) | 3 | 40 | $z$ | 0.9435 | 0.9431 |
| | | 4 | 6 | $z$ | 1.5433 | 1.5438 |
| 43 | | | | $F$ | 21.90 | 21.92 |
| 42 | | 4 | 7 | $z$ | 1.4221 | 1.4224 |
| | | 4 | 8 | $z$ | 1.3332 | 1.3333 |
| | | 5 | 6 | $z$ | 1.5177 | 1.5175 |
| | | 5 | 19 | $z$ | 0.9442 | 0.9452 |
| 43 | | | | $F$ | 6.61 | 6.62 |
| 42 | | 8 | 5 | $z$ | 1.6596 | 1.6598 |
| | | 8 | 29 | $z$ | 0.7679 | 0.7669 |
| | | 12 | 21 | $z$ | 0.7735 | 0.7734 |
| | | 24 | 5 | $z$ | 1.6123 | 1.6121 |
| | | 24 | 6 | $z$ | 1.4134 | 1.4136 |
| | | 24 | 8 | $z$ | 1.1662 | 1.1659 |
| | | 24 | 19 | $z$ | 0.7277 | 0.7279 |
| | | 24 | 21 | $z$ | 0.6964 | 0.6965 |
| | | 24 | 120 | $z$ | 0.4380 | 0.4381 |
| | | $\infty$ | 3 | $z$ | 2.4081 | 2.4080 |
| 42 | | $\infty$ | 60 | $z$ | 0.3198 | 0.3184 |
| | | $\infty$ | 120 | $z$ | 0.2199 | 0.2170 |
| 43 | | | | $F$ | 1.56 | 1.54 |

These errors were found chiefly by differencing, using comparison with other tables and recalculation where differencing was inadequate or inconclusive. It is probable that no other tabular value differs from the true value (as distinct from the "correct" tabular value) by more than unity in the last figure. However, there are a number of rounding errors, that is, tabular entries which differ from the correct tabular value by unity in the last figure, the true value lying between the two. This means that the amount rounded off is between 0.5 and 1.0 in absolute value, rather than between 0 and 0.5.

In addition, among the values of chi-square derived from Colcord and Deming's table of $z$, there are the following three errata:

| Page | $p$ | $n$ | *For* | *Read* |
|---|---|---|---|---|
| 33 | .001 | 3 | 16.268 | 16.266 |
| | | 4 | 18.465 | 18.467 |
| | | 5 | 20.517 | 20.515 |

The first of these involves a rounding error in $z$. The second and third arise because 4D in $z$ is here insufficient for 5S in chi-square.

Some explanation and comment as to the origin of these tables may prove helpful. As the tables for $p = 0.2$ were first published in F & Y, it is appropriate to remark that the 339 entries were calculated variously, 118 by interpolation in KARL PEARSON's "Tables of the Incomplete Beta Function" and "Tables of the Incomplete Gamma Function," 25 from R. A. Fisher's values of chi-square, 88 by direct calculation, and 108 (corresponding to most values of $n_2$ greater than 12) by harmonic interpolation in $z$.

Tables for $p = 0.1$ have been given for F by M. MERRINGTON & C. M. THOMPSON[3] and for $z$ and $F$ by V. G. PANSE & G. R. AYACHIT.[4] FISHER &

YATES mention both these sources. There are 41 entries in which $z$ according to F & Y differs by unity from that given by P & A. Examination of these cases shows that F & Y derived their values of $z$ from M & T. This partly explains the disagreement, because P & A derived their values of $z$ from Thompson's table, and inevitable rounding errors in the two source tables sometimes lead to different 4D values of $z$. In addition, the disagreement is explained by the inadequacy of Thompson's table for 4D in $z$, and by the occurrence of a number of rounding errors in deriving P & A's table. No rounding error was discovered in F & Y. The single gross error listed above is obviously typographical.

The statement by F & Y that their table is "based on the tables [by] Thompson" is thus shown to be misleading. In fact, contrary to Egon Pearson's statement in the introductory note thereto, the M & T table of $F$ cannot be derived from Thompson's table, as Thompson's values are sometimes insufficient for the accuracy to which M & T give F. For the same reason, F & Y cannot have derived their values of $z$ from Thompson's table. Surely also F & Y took their values of $F$ from M & T: there is no discrepancy between the two, and F & Y retain an even figure in the second decimal in every one of the five cases in which M & T give 50 in the third and fourth decimals.

The tables for $p = 0.05$ and 0.01 are entirely free of gross error for both $z$ and $F$, but there are a few rounding errors.

The table of $z$ for $p = 0.001$ as given by F & Y disagrees with that originally published by COLCORD & DEMING in ten entries. For $n_1 = n_2 = 1$, C & D gave 6.4577. When calculating the table for $p = 0.2$, I discovered this error and communicated it to Fisher. Thus F & Y give the correct value 6.4562. However, it may be noted that Fisher's "Statistical Methods for Research Workers," which included this table for the first time in the sixth edition, has repeated this error in all subsequent editions, all of which have appeared since this error was known. A second error occurred at $n_1 = 12$, $n_2 = 2$, for which C & D gave 3.4537. F & Y correctly give 3.4536, but Fisher's "Statistical Methods . . ." has always given the incorrect value.

The remaining eight differences all involve values for $n_2 = 120$, as follows:

| $n_1$ | C & D | F & Y | Correct |
|---|---|---|---|
| 1 | 1.2159 | 1.2158 | 1.2159,37 |
| 2 | 0.9948 | 0.9952 | 0.9953,81 |
| 3 | 0.8783 | 0.8773 | 0.8773,20 |
| 5 | 0.7425 | 0.7426 | 0.7425,80 |
| 6 | 0.6983 | 0.6986 | 0.6985,86 |
| 8 | 0.6329 | 0.6338 | 0.6337,39 |
| 24 | 0.4369 | 0.4380 | 0.4381,28 |
| $\infty$ | 0.2224 | 0.2199 | 0.2169,64 |

It appears that F & Y gave correct values for $n_1 = 3$, 5, and 6, improved values for $n_1 = 2$, 8, 24, and $\infty$, and replaced a correct value by an erroneous one for $n_1 = 1$. The source of F & Y's values is unknown (Yates, private correspondence). As Fisher's "Statistical Methods . . ." gives $n_2 = 1$ (1) 30, 60, $\infty$, it seems reasonably certain that the values for $n_2 = 40$ and 120 were filled in (when F & Y decided to include them) without reference to C & D.

I have tried to reproduce F & Y's values by several schemes of interpolation and approximation without success.

The table of $F$ for $p = 0.001$ was prepared from the table of $z$ by W. L. STEVENS and incorporates some of its errors. However, many of the errors in $z$ are too small to affect the values of $F$, generally to 2D. Also, the large error at $n_1 = 3$, $n_2 = 1$, is not found in the table of $F$ because Stevens calculated *de novo* all the values of $F$ for $n_2 = 1$ to 6S. This seemed the most satisfactory way of handling the problem of significant figures for these values, the 4D values of $z$ being here sufficient for only 4S in F.

A further observation is that the approximation formulas for use when $n_1$ and $n_2$ are both large are due to F & Y, not to the calculator(s) of the table to which any such formula is appended.

Lastly, the tables of $F$ by MERRINGTON & THOMPSON have received some scrutiny, though no thorough test has been made. Therefore the situation is not entirely clear, but it should not be thought that M & T are uniformly dependable in the last decimal tabulated, there being the following three errata, all for $p = 0.01$:

| $n_1$ | $n_2$ | *For* | *Read* |
|---|---|---|---|
| 3 | 120 | 3.9493 | 3.9491 |
| 8 | 1 | 5981.6 | 5981.1 |
| $\infty$ | 2 | 99.501 | 99.499 |

H. W. NORTON

University of Illinois
Urbana, Illinois

[1] C. M. THOMPSON, "Tables of percentage points of the incomplete beta-function and of the chi-square distribution," *Biometrika*, v. 32, 1941, p. 151–181, 187–191.

[2] C. G. COLCORD & L. S. DEMING, "The one-tenth percent level of '$z$,'" *Sankhya*, v. 2, 1935, p. 423–424.

[3] M. MERRINGTON & C. M. THOMPSON, "Tables of percentage points of the inverted Beta (F) distribution," *Biometrika*, v. 33, 1943, p. 73–88.

[4] V. G. PANSE & G. R. AYACHIT, "Ten percent probability of $z$ and the variance ratio," *Indian Jn. Agricultural Science*, v. 14, 1944, p. 244–247.

**200.**—G. INGHIRAMI, *Table des Nombres Premiers et de la Décomposition des Nombres de* 1 *á* 100000. 1919.

Supplementary to the list of errata in D. H. LEHMER, *Guide to Tables in the Theory of Numbers*, 1941, pp. 150–151, the following may be noted:

For 4 primes a dot (.) is not clearly printed: 69467, 69473, 69481, 69557. For 8 numbers on p. 25 corrections are here noted:

| Number | *For* | *Read* | Number | *For* | *Read* |
|---|---|---|---|---|---|
| 67069 | 4 | 47 | 68593 | | 7 |
| 68359 | 19 | 197 | 68773 | 9 | 97 |
| 68363 | 13 | 137 | 68873 | | 7 |
| 68573 | | 47 | 69169 | 2 3 | 263 |

R. C. ARCHIBALD

Brown Univ.
Providence, R. I.

201.—K. L. Nielsen & L. Goldstein, "An algorithm for least squares," *Jn. Math. Phys.*, v. 26, 1947, p. 120–132.

*P.* 123, $m = 35$ *for* $A_{55} = 488447.843200$ *read* 488447.843265
*P.* 124, $m = 85$ *for* $A_{55} = 102214274.780204$ *read* 102214274.782041
*P.* 124, $m = 90$ *for* $A_{55} = 144092594.780204$ *read* 144092594.782041

P. G. Guest

Univ. of Sydney
Sydney, Australia

202.—I. M. Vinogradov & N. G. Chetaev, *Tablitsy Znachenii Funktsii Besselia ot mnimogo Argumenta.* Moscow, Leningrad, 1950.

On pages III, V, 203–403, and on the spine, there are 408 errors in statements as to functions tabulated, namely: $J_{\frac{1}{3}}(ix)$ and $J_{-\frac{1}{3}}(ix)$. The correct functions are $i^{-\frac{1}{3}}J_{\frac{1}{3}}(ix) = I_{\frac{1}{3}}(x)$ and $i^{\frac{1}{3}}J_{-\frac{1}{3}}(ix) = I_{-\frac{1}{3}}(x)$.

R. C. Archibald

Brown University
Providence, R. I.

## UNPUBLISHED MATHEMATICAL TABLES

136[F].—A. Ferrier. *Factorization of* $n! \pm \alpha$. Photocopy of 4 manuscript pages. Deposited in the UMT File.

Two pages of tables give the complete decomposition of $n! \pm \alpha$ for $n = 7(1)15$, $\alpha = 2(1)20$ together with 13 other miscellaneous examples.

A. Ferrier

Collège de Cusset
Allier, France

137[F].—A. Ferrier. *Table of Factors of* $2^n - 1$. Photocopy of 5 manuscript pages. Deposited in the UMT File.

Two pages of tables give the latest information on the factors of $2^n - 1$, $n = 3(2)499$.

A. Ferrier

Collège de Cusset
Allier, France

138[F].—R. F. Johnson. *Tables of Products of Powers of Small Primes.* Tabulated from punched cards. Deposited in the UMT File.

There are two tables of

$$N = 2^{\alpha}3^{\beta}5^{\gamma}7^{\delta}$$

for $\alpha = 0(1)11$; $\beta = 0(1)8$; $\gamma = 0(1)5$; $\delta = 0(1)4$. The first table is arranged lexicographically by $\alpha, \beta, \gamma, \delta$. The second is arranged in increasing order of $N$. Each table contains 3240 values of $N$ range between 1 and 100818950400000. The table is intended to facilitate the design of gear trains.

R. F. Johnson

Northrop Aircraft, Inc.
Harthorne, California

**139[F].**—L. Poletti. *List of Primes of the 16th Million.* 8 pages typewritten manuscript. Deposited in the UMT File.

This list of primes ranges between 14984987 and 15105063 and contains 7277 primes.

**140[I, K].**—P. G. Guest. *Tables of Certain Functions Occurring in the Fitting of Polynomials to Equally-Spaced Observations.* Mimeographed Manuscript, 12 p. Deposited in the UMT File.

Table 1 gives the coefficients $\beta_{jk}$ in

$$\xi_j(x) = \sum_{k=0}^{j} \beta_{jk} x^k,$$

where $\xi_j(x)$ is the usual orthogonal polynomial[1] normalized so that $\beta_{kk} = 1$.

Table 2 gives the coefficients $\lambda_{jk}$ proportional to $\beta_{jk}$ such that

$$\sum_{k=0}^{j} \lambda_{jk} x^k$$

takes on the least possible integer values when $x$ ranges over the points of observations. These latter are $n$ equally-spaced points a unit distance apart having the origin as center. In both tables $n = 6(1)104$. In Table 1 $k \leq 4$ and in Table 2 $k \leq 5$. In Table 1 the sums $S_{jj}$ of the squares of the values of $\xi_j$ at the points of observation are given. Exact values are given in every case.

P. G. Guest

University of Sydney
Sydney, Australia

[1] Compare the table of Anderson & Houseman [*MTAC*, v. 1, p. 148–150].

**141[L].**—S. R. Brinkley, Jr., H. E. Edwards, & R. W. Smith, Jr., *Table of the Temperature Distribuiton Function for Heat Exchange Between a Fluid and a Porous Solid.* 141 leaves. Ms. in possession of authors, U. S. Bureau of Mines, Pittsburgh, Pa.

The function,

$$\phi_0(x,y) = e^x \int_0^x dt\, e^{-t} I_0(2\sqrt{yt}),$$

is a solution of the hyperbolic differential equation, $\partial^2\phi/\partial x \partial y = \phi$ satisfying the boundary conditions $\phi_0(o,y) = 0$, $\phi_0(x,0) = e^x - 1$. It appears in the theory of non-steady heat exchange between a fluid and a porous solid,[1] and also in the theory of ion exchange columns.[2] The present table gives $e^{-(x+y)}$ $\phi_0(x,y)$ for

$$x = y = 0(0.1)\ 5(0.2)\ 10(0.5)\ 20(1)\ 50(2)\ 100(5)\ 200(10)\ 500,\ 6D.$$

The calculations were carried out on an IBM Card Programmed Electronic Calculator. The present table is an extension of a much smaller table that has previously been announced (*MTAC*, v. 2, p. 221). A copy of the table will be made available on loan on application to the authors.

[1] A. Anzelius, *Zeit. angew. Math. Mech.*, v. 6, 1926, p. 291–294. T. E. W. Schumann, Franklin Inst., *Jn.*, v. 208, 1929, p. 405–416. S. R. Brinkley, Jr., *Jn. App. Phys.*, v. 18, 1947, p. 582–585.
[2] H. C. Thomas, Amer. Chem. Soc., *Jn.*, v. 66, 1944, p. 1664–1666.

**142[L].**—K. HIGA, *Table of* $\int_0^\infty u^{-1} \exp\{-(\lambda u + u^{-2})\}du$. One page typewritten manuscript. Deposited in the UMT FILE.

The table is for $\lambda = .01,.012(.004).2(.1)1(.5)10$. The values are given to 3S.

L. A. AROIAN

Hughes Aircraft Co.
Culver City, California

**143[L].**—Y. L. LUKE. *Tables of an Incomplete Bessel Function.* 13 pages photostat of manuscript tables. Deposited in the UMT FILE.

The tables refer to the function

$$j_n(\mu,\theta) = \int_0^\theta \exp\{i\mu \cos\phi\} \cos n\phi d\phi.$$

Values are given to 9D for

$$\begin{aligned} n &= 0, 1, 2 \\ \cos\theta &= -.2(.1).9 \\ &= 49\omega/51, \omega = 0(.04).52. \end{aligned}$$

There are also auxiliary tables. The tables are intended to be applied to aerodynamic flutter calculations with Mach number .7.

Y. L. LUKE

Midwest Research Institute
Kansas City, Missouri

## AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 415 South Building, National Bureau of Standards, Washington 25, D. C.

### TECHNICAL DEVELOPMENTS

# Fundamental Concepts of the Digital Differential Analyzer Method of Computation

**Introduction.**—Two fundamentally different approaches have been developed in using machines as aids to calculating. These have come to be known as analog and digital approaches. There have been many definitions given for the two systems but the most common ones differentiate between the use of physical quantities and numbers to perform the required automatic calculations.

In solving problems where addition, subtraction, division and multiplication are clearly indicated by the numerical nature of the problem and the data, a digital machine for computation is appropriate.

When problems have involved calculus methods, as, for instance, in the solution of differential equations, the analog computer has often been used,

as the process of integration seems, psychologically at least, to be more aptly handled by analog devices. These devices have been mechanical or electronic integrators. However, the actual process of integration, if one considers the numerical basis for its origin is, in a sense, a numerical additive process. Thus, digital computers "integrate" by successive additions.

By bridging the gap between these two approaches, a new series of instruments for computation is possible.

The method of computation used in a digital differential analyzer resulted from the adoption of a new point of view. Considering the operations involved in the solution of differential equations, it is possible, with this new approach, to obtain many of the advantages of a digital computer and also the essential advantages of an analog differential analyzer. The result is a different type of digital "logic" from that used in the general purpose digital computers.

The advantages gained by the new method in solving ordinary differential equations of any type are:

1. Ease of preparing problems—arising from the use of analog differential analyzer methods instead of numerical methods in the coding process.
2. Increase in computation speed over equivalent general purpose digital computer approaches and equality in speed to some analog methods.
3. Increase in accuracy over analog differential-analyzer procedures.
4. Repeatability and ease of error analysis inherent in the digital method.
5. Small size—In certain embodiments the digital differential analyzer can be much smaller, have fewer tubes and components, weigh and cost less than analog differential analyzers or any of the general purpose digital computers. This is particularly true when the number of integrators needed to solve the equations becomes large.

**Review of Analog Differential Analyzer Theory.**—There are two different ways of explaining the digital differential analyzer method. The first is a qualitative explanation which follows the analog viewpoint and points out the first advantage. The second is a quantitative numerical explanation which shows the error analysis possibilities and the successive-additions method of integration which actually takes place.

To appreciate the first explanation, it is necessary to review the principles of the analog differential analyzer. In solving an equation such as,

$$\frac{d^2w}{dt^2} - w\frac{dw}{dt} - wt = 0 \tag{1}$$

the analog differential analyzer represents the variables $w$, $t$, $\frac{dw}{dt}$ etc., by mechanical rotations of shafts or by variations of voltages in electronic circuits. The rates of shaft rotations or of changes in voltages are always proportional to the rates of change of the variables.

Integration is accomplished by a mechanical wheel and disc integrator or an operational amplifier used as an integrator. Other mathematical operations such as multiplication and addition of variables in the equation are performed either by integrators or other devices, mechanical and electronic.

Integrators and other units are interconnected in such a manner as to produce an analog of the differential equation. The set is "driven" by a

single shaft or voltage representing the independent variable ($t$ of Equation 1). The $w$ or dependent variable shaft or voltage varies in accordance with the actual solution to the equation as $t$ varies. For a given set of initial conditions, a solution to Equation 1, $w = f(t)$, is produced as either a graph or a set of tabular values of $w$ as a function of $t$.

The integrator is the key to the machine's operation, all other units being straightforward in comparison. It may be regarded as a "black box" with two inputs and one output shown schematically in Figure 1.

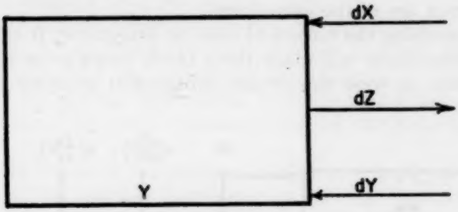



FIG. 1.

The inputs[1] $dx$ and $dy$ are the rates of change of some $x$ and $y$ variables in an equation as represented physically by shafts rotating or voltages changing. The differential notation is used because the same $dt$ is inherently used throughout the machine. The inputs and outputs are related by the integrator Equation 2.

$$dz = KYdx, \tag{2}$$

where $Y = S\,dy$. The constant $K$ is determined by the physical properties of the integrator.

In the mechanical integrator the $dy$ input causes a worm gear to move a small disc across the surface of a large wheel (see Figure 2) such that its distance from the center of the wheel is $Y$. The $dx$ input turns the wheel and friction causes the disc to rotate at a speed $dz$.

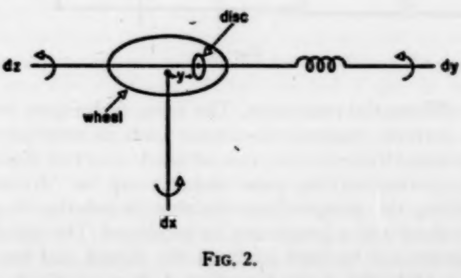



FIG. 2.

In the electronic integrator the $dy$ input is a varying voltage, the $dx$ input is always time and the $z$ voltage is produced by using the integrating characteristics of capacitors in connection with a feedback amplifier to

produce linearity. It should be noted that to interconnect integrators it is necessary that the inputs and outputs all be of the same form.

**Qualitative Explanation of the Digital Integrator.**—The digital integrator is the heart of the new type of computer, the digital differential analyzer, and may be visualized as a black box with the same schematic (Figure 1) and the same equation relating its inputs and output (Equation 2). The $dx$, $dy$, and $dz$ variables are represented by pulse rates, i.e., the rates of occurrence of streams of electronic pulses entering or leaving the integrator. As stated before the equation relating these pulse rates is still Equation 2, and the inputs and output are of the same form.

Without describing the nature of such an integrator, it can be seen that the two properties above will allow these black boxes to be intercoupled, in the same manner as were the analog differential analyzer integrators, to

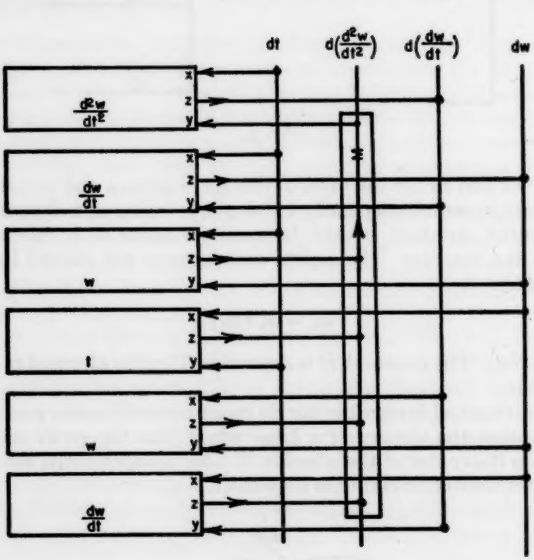



FIG. 3.

solve ordinary differential equations. The same techniques of intercoupling integrators to perform various operations such as multiplication, scaling function generation, division, etc., can be used. A set of digital integrators intercoupled by wires carrying pulse streams can be "driven" by a pulse source representing the independent variable. A solution is produced as a set of tabular values and a graph can be produced. The same schematic or connection diagram can be used for both the digital and the analog differential analyzers. Only the $K$ in Equation 2 changes. Such a schematic is illustrated in Figure 3 for the solution of Equation 1.

Two distinct advantages of the digital over the analog integrator in addition to increased accuracy should be noted at this point. It will be

observed that Figure 3 contains no adders. The terms

$$w\,dt,\ t\,dw,\ \left(\frac{dw}{dt}\right) dw, \text{ and } w\,d\left(\frac{dw}{dt}\right)$$

created by the lower four integrators would in the analog machines have to be added in extra units, known as adders, to form $d\left(w\frac{dw}{dt} + w\,t\right) = d\left(\frac{d^2w}{dt^2}\right)$ to be fed back into the "$Y$" input of the upper integrator. The addition is indicated on the diagram by a box with $\sum$ sign.

Since the outputs of these integrators in the digital case are pulse streams, they may be mixed together directly and sent into the same input, provided, of course, that the pulses do not coincide in time. In several embodiments of the digital differential analyzer this is the case. If time coincidence does occur, it is only necessary to delay one pulse with respect to another.

The other advantage is the superior ability of the digital integrator to receive the output of another integrator at its "*dx*" input. This greatly facilitates multiplication, division and the solution of non-linear equations without the use of special devices.

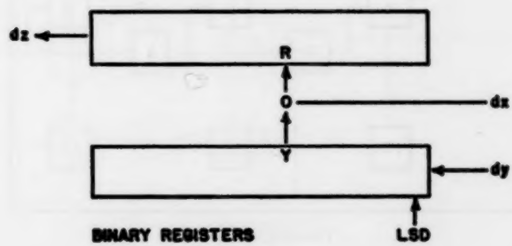



FIG. 4.

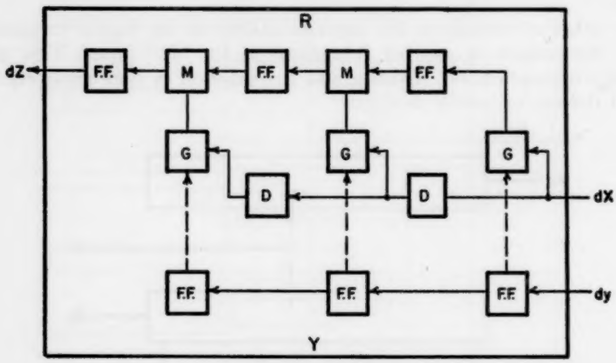
**Embodiments of the Digital Integrator.**—The "contents" of the black box digital integrator may take many physical forms and still have the external properties described. All of the forms so far devised have one common property. Two numbers appear within the box and may be designated as a coupled pair. These two numbers, always labelled $Y$ and $R$, may appear in any one of several physical forms. Two specific examples are: Numbers stored in vacuum tube registers made up of two-stable-state devices, and numbers appearing in pulse form on a cathode ray tube screen. The numbers may be of any length and in any number base system. For convenience they will usually be represented in this paper schematically as appearing in two registers as binary numbers. (See Figure 4.) Some other methods of storing the $Y$ and $R$ numbers are: relays, mechanical registers as on desk calculators, magnetic tapes or drums, and mercury or other delay lines. Each of the storage media must be capable of changing the numbers digitally by the receipt of information at the inputs to the box.

In the integrator diagram in Figure 4, the $Y$ register acts as a counter

when receiving $dy$ pulses and in a sense integrates the $dy$ pulse rate to produce the number $Y$. The $dx$ pulses are treated as instructions to transfer in an additive manner the number $Y$ into the $R$ register without removing $Y$ from the $Y$ register. If the $R$ register contained some previous $R$ before the transfer, it contains $R + Y$ after the transfer.

The $R$ register will of course overflow after a certain number of transfers. Each time it does so, a pulse is transmitted from the integrator as a $dz$ pulse. The $Y$ and $R$ registers have the same length and capacity and, if binary registers are used, the capacity is $2^N$, where $N$ is the number of binary stages in each register.

By qualitative analysis of the relations between the variables, it may be seen that Equation 2, $dz = KY\,dx$, does hold for the integrator, where $K = 1/2^N$ and $Y$ is regarded as an integer, provided that $Y$ remains constant. In other words if $Y = 1$ the output rate $dz$ will be $1/2^N\,dx$, since it requires




Transfer method.

FIG. 5.

$2^N$ additions of $Y$ to $R$ to cause an overflow. If $Y = 2^N$, or the register is filled to capacity, an $R$ overflow or $dz$ pulse will occur for every $dx$ pulse. In this case, $dz = dx$. In general, $dz$ is certainly proportional to $dx$ for a constant $Y$ and is proportional to $Y/2^N$ for constant $dx$.

Two fundamentally different ways exist (as well as combinations of the two) to cause the transfer of $Y$ to $R$ to take place. In the one described above, called the transfer method, a single pulse at the $dx$ input caused the entire $Y$ number to be added into $R$. In the second system a large number of $dx$ pulses are required to transfer $Y$ to $R$. The $Y$ number may change during this transfer process so that the number of stages required in the register for the same accuracy is larger. This method is called the sieve method.

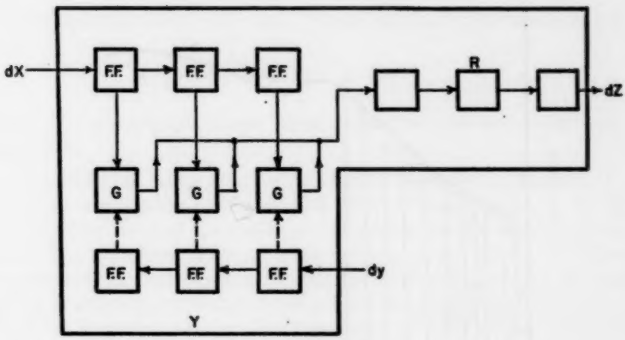
Figures 5 and 6 illustrate electronic methods of causing the two types of transfer. There are, of course, many other electronic ways of effecting the transfer.

In Figure 5 the additive transfer of $Y$ into $R$ by a single $dx$ pulse is accomplished by transmitting the $dx$ pulse into successive gates and delays. The

gates are controlled (dashed lines) by the $Y$ register flip-flops, or two-state devices. If a flip-flop is in its "1" state (binary digit one at that digit position), its gate is "open" and the $dx$ pulse passes up to one of the $R$ register flip-flops causing it to trigger. If the $Y$ flip-flop is in its "0" state (binary digit zero), the gate is closed and the pulse does not pass through.

If an $R$ flip-flop triggers from "1" to "0" it transmits a "carry" pulse to the next flip-flop to the left. The $dx$ pulse in the meantime is being delayed through $D$, and if it passes the next gate it will arrive at the pulse mixer $M$ non-coincident with the carry pulse from the $R$ flip-flop. Any carries from the left $R$ flip-flop represent overflow pulses and are transmitted to the $dz$ output.

The sieve method of Figure 6 requires $2^N$ $dx$ pulses to transfer $Y$ into $R$. A third register is used to distribute the $dx$ pulses through the gates controlled by the $Y$ flip-flops in such a manner that when the outputs from each gate are mixed they are non-coincident and can be accumulated in $R$. The entire operation resembles the action of sieving the $dx$ pulses through the gates.




Sieve method.

FIG. 6.

Carry pulses are taken from the third register flip-flops at different times to feed to the gates and to trigger the next flip-flop to the right. This means that each set of pulses reaching the gates as the operation proceeds from left to right is anticoincident with all preceding sets and equal to half the adjacent left hand set. The effect of $2^N$ ($2^3$ in case shown) $dx$ pulses for constant $Y$ will be to transfer $Y$ into $R$.

When other storage methods are used, the electronic operations change. For instance with magnetic drum storage, the two numbers are stored in two parallel channels, and the digits of $Y$ and $R$ appear at magnetic read heads one digit at a time in serial fashion. The addition of $Y$ to $R$ is then that of time-serial binary addition. The same thing would be true of any serial or delay type of number storage.

**Quantitative Explanation of the Digital Integrator.**—The second or quantitative explanation of the digital differential analyzer method will be covered rigorously in another paper. Briefly, the successive addition process of multiplying an ordinate of a curve, $Y \equiv f(x)$ (see Figure 7), by a $\Delta x$

increment and adding the resulting products to get the area under a curve is really being carried out in an integrator.

If the *R* register were of unlimited length and each *dx* pulse were assumed to have a value of 1, then the successive additions of *Y* to *R* would produce a number similar to the sum of the area of the rectangles under the curve of Figure 7 (assuming the *Y* values to be correct). Since the *R* register is broken off and *dz* pulses transmitted, it can be seen the sum of these *dz* pulses will be in error from the rectangular areas by the remainder *R* in the *R* register. The total error consists of this roundoff error plus the truncation error difference between the true curve and the rectangles. Automatic corrections of various electronic types can be and have been made for both of these errors. In the future mathematical paper it will be demonstrated that the total truncation and roundoff error for many equations will not exceed the two least significant binary digits of a *Y* register.

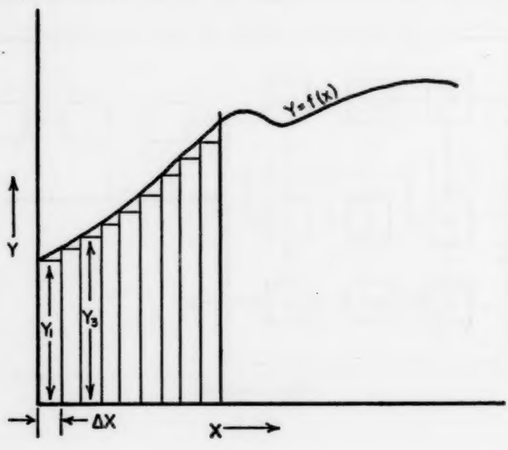



FIG. 7.

**Further Advantages of the Digital Integrator.**—In the foregoing discussion "black boxes" comprising digital integrators which have two fundamental properties were described. Ordinarily, to obtain 50 "black boxes" it would be necessary to use 50 times the equipment required for one box. This is certainly true in case of the mechanical and electronic integrators. However, where the integrators consist merely of paired numbers operating on each other in accordance with the methods already described, it becomes possible to time share operational circuits among all of the number pairs if they are stored in a serial or delay type of memory and the integrators are strung out in a line timewise.

One set of electronic circuits can operate on all integrators in sequence, or rather on all paired numbers in sequence. An integrator, as such, does not really exist when such a system is used. In a scheme like this it is easily seen that no coincidence problems exist, since no two integrator outputs

occur simultaneously. It will also be seen that the amount of equipment does not increase linearly with the number of integrators and that beyond a certain point the digital differential analyzer is smaller and involves less components than the analog.machine.

The problems of handling the signs of the variables and their derivatives and of the scales or scale factors for a problem will also be covered in detail in a future paper. They are similiar to analog sign and scale problems except that special provisions must be made for handling signs, and scaling takes place digitally.

The writer wishes to express thanks to the following men who furnished the ideas for much of the material this article covered: D. E. Eckdahl, H. H. Sarkissian, I. S. Reed, C. Isborn, W. Dobbins, F. G. Steele, B. T. Wilson, J. Donan, J. Matlago, and A. E. Wolfe.

R. E. Sprague

Computer Research Corporation
Torrence, California

[1] V. Bush and others use the $x$, $y$, and $z = y\,dx$ notation for inputs and outputs.

## Bibliography Z-XVIII

1. Anon., "Computing machines," *Mechanical Engineering*, v. 73, Apr. 1951, p. 325–327.

The MADDIDA (Magnetic Drum Digital Differential Analyzer), a new small electronic computer built by Northrup Aircraft, Inc., is briefly described, including some of the main specifications and the uses for this type of computer. The machine is capable of solving many types of differential equations or sets of such equations. The machine is being manufactured for general use in science and industry, at a relatively low cost. It is a 29 binary digit machine and adds binary digits at a rate of 100,000 per second. A big advantage of the machine is that differential equations can be solved without reducing them first to difference equations.

A survey of the Federal Computer Program is also included in the article and a list of many of the analog and digital computers either being used or being constructed at various institutions in the United States.

Donald Larson

NBSCL

2. Anon., "High-speed analog to digital converter," *Review Sci. Instr.*, v. 22, July 1951, p. 544.

Expository article.

3. S. Gill, "The diagnosis of mistakes in programmes on the EDSAC," R. Soc., London, *Proc.*, v. 206A, 1951, p. 538–554.

This paper discusses in detail two methods which have been used to diagnose errors in programming on the EDSAC. The "Blocking Order" routine prints the contents of any given location whenever the blocking order is obeyed, and then returns the control to the main routine immediately after the blocking order. This is useful in investigating arithmetical failures.

The "step by step" routine prints operation symbols as they occur during the run of the problem, thereby giving a compact representation of the progress of the computation. This is of use in investigating order failures. The general ideas of these routines are readily adaptable to other high-speed computers; indeed on a four address machine these routines can be made much more simple and flexible.

OTTO STEINER

NBSCL

4. OFFICE OF NAVAL RESEARCH, *Digital Computer Newsletter*, v. 3. no. 3, Oct. 1951, 5 pages.

The present status of the following digital computer projects is treated briefly in this number:

1. The Circle Computer
2. Naval Proving Ground Calculators
3. Aberdeen Proving Ground Computers
4. Electronic Computer Corporation Computers
5. The ORDVAC
6. The SEAC
7. The SWAC
8. The Raytheon Computer
9. The University of Toronto Electronic Computer UTEC
10. The Ferranti Computer at Manchester University, England

Component Developments

1. The Computer Research Corporation
2. Physical Research Laboratories Computer Development

5. A. G. RATZ & V. G. SMITH, "A method of gating for parallel computers," AIEE, *Trans.*, v. 70, 1951, p. 424.

The problem is to set a slave register into the same state as a master register, where each register is a row of Eccles-Jordan flip-flop circuits and both are operated at the same supply voltages. A pair of diodes connect each master flip-flop to its corresponding slave. The cathodes of the diodes connect to the plates of the master, and the plates of the diodes connect to the opposite grids of the slave. The diodes are normally nonconducting, but transfer is effected by a pulse which lowers the plate supply voltage of the master register so that the lower voltage plate of a master flip-flop forces down the voltage of the opposite grid of its slave through conduction of the connecting diode.

R. D. ELBOURN

NBSCML

6. J. R. TILLMAN, "Transition of an Eccles-Jordan circuit," *Wireless Engineer*, v. 28, Apr. 1951, p. 101–110.

Waveforms of plate voltage during transition are computed for a conventional low-tube bistable circuit without "speed-up" capacitors, triggered on the control grids. Both linear and parabolic plate current characteristics

are considered. Particular attention is given to initial conditions near the threshold of instability and to triggering pulses so short that the loop gain is raised just above unity so that positive feedback can complete the transition. Twenty millimicroseconds can suffice for triggering; however, reliability and practical tolerances may require more stable initial conditions and stronger triggering.

R. D. Elbourn

NBSCML

7. H. C. Tuttle, "Machine brains dissected at computing conference," *Steel*, v. 128, Apr. 2, 1951, p. 112–114.

This article discusses in a non-technical way some of the uses to which digital computing machines may be put in solving the many practical problems in industry, especially in the design of equipment. Some papers are mentioned in the article which were given at a conference on automatic computing machinery at Wayne University on Mar. 27–28, 1951. These papers apply mainly to the business man's interest.

Donald Larson

NBSCL

8. M. V. Wilkes, D. J. Wheeler, & S. Gill, ***The Preparation of Programs for an Electronic Digital Computer***. 167 p. Addison-Wesley Press, Inc., Cambridge, Mass., 1951. 22.9 × 15.2 cm. Price $6.00.

Although this book is almost exclusively devoted to programming for the EDSAC (Electronic Delay Storage Automatic Computer) at the University of Cambridge, it contains a great many items of interest for persons engaged in programming for other machines of similar type. It is of particular interest to persons responsible for putting such high-speed digital computers into business on a production basis. Emphasis is placed, in this book, on minimizing coding time and reducing the incidence of errors by the use of an adequate library of prefabricated subroutines. Such subroutines may be used as building blocks for constructing almost the complete program. By thus economizing on programming time it is possible to exploit to full advantage the capabilities of a limited programming staff, and it becomes feasible for the digital computer to handle problems requiring relatively small amounts of operating time.

The book opens with a Preface by the authors and a Foreword by D. R. Hartree. The text is divided into three main parts, Part I dealing with general matters of coding for EDSAC, and the remaining two parts with specific details of EDSAC subroutines. There are in addition five appendices, the first of which gives the keyboard code, the corresponding teleprinter symbols, the code as punched on tape, and the numerical interpretations of the keyboard code symbols. The remaining four appendices give further coding details. Throughout the text the explanations are buoyed by means of numerous detailed examples and illustrations. There is also a bibliography of publications on EDSAC at the end of Chapter 2.

The opening chapter briefly discusses large scale automatic digital computing machinery in general terms, and the various kinds of codes, but quickly settles down to a description of the EDSAC, its "single address" order code, and the use of this code.[1]

The second chapter is fundamental and deals with the process by which information is read from tape and placed in the store (memory). This input process is accomplished with the aid of the "initial orders." These are a sequence of built-in instructions, activated when the start button is pressed, which direct the input process. A detailed listing of these initial orders together with accompanying explanation is given in Appendix B. The input process may be further controlled or modified in two ways: (a) by means of "code letters" (used to terminate orders), and (b) by the use of so called "control combinations," i.e., punched groups of symbols appropriately interspersed among the orders on the input tape. These two kinds of indications are detected and interpreted by the initial orders; when properly chosen and placed, they serve to make the input process fully flexible. In particular, the use of this system makes it possible to code subroutines without regard to how they are to be integrated into the complete routine nor where they are to be placed in the store, and to file them in the library in the form of short lengths of tape. Then, when needed, these subroutines are copied mechanically onto the program tape, and by means of the system of initial orders, code letters, and control combinations, they are automatically integrated during input into the complete routine. The control combinations in most common use are given in the text. A further list is given in Appendix C.

Chapter 3 presents a brief explanation of the method used on EDSAC for entering and leaving a subroutine, together with methods for inserting parameters into subroutines.

Chapter 4 contains a general description of the EDSAC library of subroutines. In particular, "assembly subroutines" are described for putting the various components of a complete program together in the store. These are designed to relieve the programmer of the mechanical tasks of deciding where the master routine and each subroutine are to go in the store, and of inserting the necessary orders for linking the components together. This chapter also describes four library subroutines for integrating ordinary differential equations (not necessarily linear), subroutines for operating on complex numbers, for floating point operations, and others.

In the fifth chapter, entitled "Pitfalls," are discussed some of the more common types of errors that arise in program preparation and ways of reducing or eliminating them. Every effort is made to eradicate errors before a problem goes on the machine. However, the authors observe that rarely does a program work right the first time it is tried, so that efficient code checking techniques on the machine become necessary. Some of these are described.

In Chapter 6 is presented a description of the auxiliary EDSAC equipment for tape punching, editing, duplicating, and comparing. Brief mention is made of EDSAC controls, of the operating organization, and of the method of storage of library subroutines.

Chapter 7, which concludes Part I presents detailed coding examples.

Part II itemizes the specifications for the library of subroutines. More precisely, corresponding to each subroutine in the library, this chapter records the vital statistics such as type of subroutine, total number of

storage locations occupied, an explanation of what the subroutine does, etc.

Finally, Part III gives the detailed programs for a selected list of subroutines.

JOSEPH H. LEVIN

NBSCL

[1] In line 5 of page 6 the order code symbol should read LD instead of LF.

## NEWS

**Commonwealth Scientific and Industrial Research Organization.**—About 200 persons from Universities throughout Australia, various Divisions and Sections of C.S.I.R.O., other government bodies including the Department of Supply, and from certain industrial and commercial firms gathered in Sydney for a Conference on Automatic Computing which was held in the Electrical Engineering Department of the University of Sydney on the 7th, 8th, and 9th of August. The conference was arranged by the Commonwealth Scientific and Industrial Research Organization which has been interested in this field of research for some years now, both because of the importance of mathematical analysis in scientific work and also because its Mathematical Instruments Section and the Computing Group in its Radiophysics Division have been developing automatic computing machines. The conference coincided with the presence in Australia of Professor D. R. HARTREE of Cambridge, who gave stimulating leadership to much of the business of the conference and set the discussion against a background of world progress which has been due in no small measure to his own work in this field.

The conference was opened by Professor JOHN MADSEN, who indicated that it would take place in two sessions, the first of which was intended primarily to emphasize the application of computing aids to industrial, commercial, and research problems. The second session would deal with the more detailed problems of numerical methods and programming and also with some engineering developments in computing equipment.

A general introduction to automatic calculating machines by Hartree was followed by a lecture and demonstration by D. M. MYERS and W. R. BLUNDEN on the C.S.I.R.O. Differential Analyser. This instrument was completed several months ago and contains ten integrators of the disc, ball, and cylinder type. Unit construction has been adopted throughout, the interconnection of units being made through step-by-step electrical transmission, providing great simplicity in setting up equations.

The afternoon sitting was opened by Hartree who explained the basic operations that take place in a high-speed automatic digital machine and the manner of controlling the machine by sequentially stored instructions in "one address" form. This was followed by an account and demonstration of the C.S.I.R.O. Mark I Electronic Digital Computer by T. PEARCEY and M. BEARD. This machine, which is now coming into service in the Radiophysics Division, uses mercury delay lines for its main store and has a capacity of 1,024 words of 20 binary digits and a pulse repetition frequency of 333 Kc/s. A magnetic drum auxiliary store is being developed for the machine.

In the first part of Session II, Hartree and Pearcey explained in some detail the problems associated with the organization of calculations for automatic machines. This led to a discussion of programming, i.e., the compilation of sets of instructions to deal with a calculation; and also to the manner in which a mathematical problem is reduced to a form suitable for programming. An account of programming for punched card and certain types of desk machines was also included. The second part of this session started with a discussion led by Pearcey on the interaction between programming and machine design and was followed by brief accounts of some new devices, including magnetic control circuits, magnetic drum storage, and electron beam tubes for decimal counting and binary switching by B. F. C. COOPER, D. L. HOLLWAY, D. M. MYERS, and C. B. SPEEDY.

Short accounts of analogue computing by Myers and of digital-analogue conversions by W. R. BLUNDEN were given, and also an electrical analogue machine for solving polynomials, recently constructed at Adelaide University by W. G. FORTE and G. A. ROSE, was described. The conference concluded with a general discussion. Demonstrations of the C.S.I.R.O. machines and of various desk and punched card machines which were exhibited by various accounting machine firms were carried on concurrently with the conference.

**NBSINA.**—On August 23–25, 1951, at the Institute for Numerical Analysis, a Symposium on Simultaneous Linear Equations and the Determination of Eigenvalues was held under the auspices of the National Bureau of Standards in cooperation with the Office of Naval Research. This was one of a series of symposia which the Bureau is holding as part of its scientific program for the year 1951 in marking the fiftieth anniversary of its establishment. The program was as follows:

| | |
|---|---|
| Thursday, August 23, 1951 | Registration |
| General session | J. H. CURTISS, *Chairman*, NBS |
| Classification of methods for solving linear equations and inverting matrices | G. FORSYTHE, NBS |
| Some problems in aerodynamics and structural engineering related to eigenvalues | R. A. FRAZER, National Physical Laboratory, Teddington, Middlesex, England |
| The geometry of some iterative methods of solving linear systems | A. S. HOUSEHOLDER, Oak Ridge National Laboratory |
| Session on Linear Equations and Inversion of Matrices | J. TODD, *Chairman*, NBS |
| Solutions of simultaneous systems of equations | A. OSTROWSKI, Universität Basle, Basle, Switzerland |
| Solutions of linear systems of equations on a relay machine | C. E. FRÖBERG, Lund, Sweden |
| Some special methods of relaxation technique | E. STIEFEL, Zurich, Switzerland |
| Errors of matrix computations | P. S. DWYER, University of Michigan |
| Friday, August 24, 1951 | |
| Session on determination of eigenvalues | J. B. ROSSER, *Chairman*, Cornell University |
| Inclusion theorems for eigenvalues | H. WIELANDT, Tübingen, Germany |
| On a general computation method for eigenvalues | G. FICHERA, Trieste, Italy |
| Variational methods for the approximation and exact computation of eigenvalues | A. WEINSTEIN, University of Maryland |
| Session on determination of eigenvalues | A. W. TUCKER, *Chairman*, Princeton University |
| Iterative methods for finding eigenvalues and eigenvectors | M. R. HESTENES, NBS and UCLA |
| New results in the perturbation theory of eigenvalue problems | F. RELLICH, Göttingen, Germany |
| Determination of eigenvalues and eigenfunctions | H. H. GOLDSTINE, IAS |
| Bounds for characteristic roots of matrices | A. T. BRAUER, University of North Carolina |
| Saturday, August 25, 1951 | |
| General Discussion of Problems | F. J. MURRAY, *Chairman*, Columbia University |

# OTHER AIDS TO COMPUTATION

## Bibliography Z–XVIII

9. Sumner Ackerman, "Precise solutions of linear simultaneous equations using a low cost analog," *Rev. Sci. Instr.*, v. 22, 1951, p. 746–748.

The author describes a simplified adjuster type of electic analog simultaneous linear equation solver, which is used to find the quasi-inverse matrix, this is applied to transform the original matrix to a quasi-diagonal form to expedite numerical solution.

The machine uses uncalibrated potentiometers to set in coefficients and determines these by a measurement process using a voltmeter. The unknowns are also controlled by potentiometers, and are adjusted to the desired solution by the operator. The operator's indication is a meter displaying an error-function proportional to the sum of the squares of the equation errors. Results are read by a voltmeter.

It is claimed that with this type of machine, the quasi-solutions required for the diagonalizing process are easier to obtain than direct solutions which require a real minimizing of the error-function.

Robert M. Walker

Watson Scientific Computing Laboratory
New York 27, N. Y.

10. Wilhelm Bader, "Auflösung von Polynomgleichungen auf elektrischem Wege," *Zeit. angew. Math. Mech.*, v. 30, 1950, p. 289–291.

A device is described for solving polynomial equations, $P(z) = 0$, by means of $n$ alternating current generators of frequency $\nu$, $2\nu$, $\cdots$, $n\nu$. The amplitude of the generator with output $k\nu$ is controlled to be $r^k$ and the real and imaginary part of the polynomial $P(z)$ are obtained in the obvious way and applied to the deflection plates of an oscilloscopic tube so that for $r = |z|$ fixed, the oscilloscope shows the locus of the values of $P(z)$. $r$ is varied by a manual control until this locus goes through zero. The argument $\theta$ for the root $z = r \exp (i\theta)$ is located by means of a magnetic "marker" mounted on the generator of the fundamental frequency which will cause a bright dot to appear on the locus, at a point corresponding to any fixed value of $\theta$. A photograph of the device appears. An electrolytic tank device for this purpose is also planned.

F. J. M.

11. Hans Bückner, "Zum Zirkeltest der Integrieranlagen," *Zeit. angew. Math. Mech.*, v. 31, 1951, p. 224–226.

The author considers two wheel and disk integrators with equations $dz_i = y_i dx$, $i = 1,2$ so connected that $y_1 = -z_2$, $y_2 = z_1$ (*). These equations yield $\ddot{z}_i + z_i = 0$, $i = 1,2$, the equations of harmonic motion. In a perfect integrator the curve in the $z_1$, $z_2$ plane would be a circle (say of radius C). He considers two problems. (i) Suppose that instead of (*), $y_1 = -z_2 - f(z_2) \equiv -\varphi(z_2)$ and $y_2 = z_1 + g(z_1) \equiv \psi(z_1)$, where $\varphi$ and $\psi$ are continuous monotonic functions in the infinite interval and $f$ and $g$ are bounded. Then he shows that the curves in the $z_1$, $z_2$ plane will be closed curves, free of double

points. (ii) The second problem considers the effects of backlash. He assumes $y_1 = -z_2 + \Delta z_2$, $y_2 = z_1 - \Delta z_1$ (**), where $\Delta z_2 = a_2 \operatorname{sgn} \dot{z}_2$, $\Delta z_1 = a_1 \operatorname{sgn} \dot{z}_1$ (except in the neighborhood of $\dot{z}_1 = 0 = \dot{z}_2$). The constants $a_i \geqslant 0$, $i = 1,2$ are due to the play in the driving mechanisms. By integration of the nonhomogeneous equations (**) the author determines that instead of closed curves in the $z_1$, $z_2$ plane, spirals are obtained. The radius of the spirals is approximately $8n(a_1 + a_2)C$, where $n$ is the number of turns. No other types of errors in mechanical integrators are considered.

K. S. Miller

New York Univ.
New York

**12.** J. B. Carroll & C. C. Bennett, "Machine shortcuts in the computation of chi-square and the contingency coefficient," *Psychometrika*, v. 15, 1950, p. 441–447.

"The methods presented here are particularly adapted to the more recent models of desk calculating machines. . . ."

**13.** G. F. Castore & W. S. Dye III, "A simplified punch card method of determining sums of squares and sums of products," *Psychometrika*, v. 14, 1949, p. 243–250.

**14.** S. Charp, "A new Fourier series harmonic analyzer," *Electrical Engineering*, v. 68, 1949, p. 1057.

The coefficients $a_n = \pi^{-1} \int_0^{2\pi} f(x) \sin nx \, dx$ and the corresponding $b_n$ are obtained from a graph of the function $f(t)$ by means of ball and roller integrators. $f(x)$ is traced by an operator from the graph and the sine and cosine are obtained by a Scotch yoke mechanism.

**15.** Lisbeth Crowell, "The airflow slide rule," *Aero Digest*, v. 59, 1949, No. 6, p. 66, 76, 78. Also, Franklin Inst. *Jn.*, v. 249, 1950, p. 328–332.

**16.** H. J. Dreyer, "Automatisches lichtelektriches Kurvenabtasten bei Integrieranlagen," *Zeit. angew. Math. Mech.*, v. 30, 1950, p. 291–292.

The device described in the title is for use in the Darmstadt differential analyzer.

**17.** E. D. Jarema, "Noise figure chart," *Electronics*, v. 23, 1950, No. 3, p. 114.

**18.** K. Krienes, "Ein Polarplanimeter zur Bestimmung des polaren Trägheitsmomentes," *Zeit. angew. Math. Mech.*, v. 30, 1950, p. 62.

A mathematical derivation of the theory for this device.

**19.** G. Lyon, "Some experience with a British A. C. network analyzer," Inst. Elec. Engrs., *Proc.*, v. 97, part II, 1950, p. 697–714. Discussion p. 714–725.

The author describes a British built post-war network analyser which uses a 500 c.p.s. supply system. The number, range, accuracy, construction

and composition of each of the various components are given. The remainder of the paper is concerned with the application of this device to a wide range of engineering problems of power network design.

20. J. L. Meriam, "Differential analyzer solution for the stresses in a rotating bell-shaped shell," Franklin Inst., *Jn.*, v. 250, 1950, p. 115–133.

The author obtains a fourth order ordinary linear differential equation for the stress in a rotating shell which consists of a portion of a torus with zero hole. This differential equation was solved on the UCLA differential analyzer. The problem was solved under various boundary conditions on the two edges. Since the equations are linear, no basic difficulty is involved in obtaining a solution from the solutions available in the differential analyzer but the author indicates useful techniques.

F. J. M.

21. Thomas M. Moore, "German missile accelerometers," *Electrical Engineering*, v. 68, 1949, p. 996–999.

These accelerometers for V-2 rockets are tied in with an integrating process to yield a specified velocity at the point where the fuel is cut off. For this purpose accelerometers based on precessing gyroscopes were inadequate. A system was devised in which an electric current from a linear inertia accelerometer was fed into an electroplating cell. After a certain prescribed amount of electricity was fed into the cell, a change in character of the electrolytic process caused a definite voltage jump. This permitted an integration process precise to .03 percent. To provide a compensation for distance covered, a further integration is necessary. The article also discusses the lateral control of these rockets.

F. J. M.

22. K. Ramsayer, "Die Funktionsrechenmaschine," *Zeit. angew. Math. Mech.*, v. 30, 1950, p. 294–295.

The author discusses the addition of a function table to a desk type calculating machine which is provided with (mechanical) storage facilities. The values are to be stored on a template and provision is to be made for linear interpolation. A similar machine was constructed in Germany during the war but is no longer available.

F. J. M.

23. Reference Sheet Section, *Electronics*, v. 23, 1950, "Buyers Guide," p. R1–R40.

This section contains 21 articles involving graphs and tables of assistance in computation for electrical engineering, for instance, one is on "vector computation," another on the "square root of a complex number." It also includes a complete index of all such articles published in *Electronics* since 1930.

24. M. G. Say, "Analogies," Inst. Elec. Engrs., *Proc.*, v. 97, 1950, part I, p. 21–22.

25. G. B. WALKER, "Factors influencing the design of a rubber model," Inst. Elec. Engrs., *Proc.*, v. 96, part II, 1949, p. 319–324. Discussion of this paper v. 97, part II, 1950, p. 439–444.

In the design of vacuum tubes a rubber membrane is often used to give a gravitational reproduction of the potential field in the tube and steel balls are used to reproduce the electrons. The author discusses the errors due to the fact that the surface of the membrane does not exactly satisfy the Laplace equation, the error due to spin of the ball around the axis normal to the surface and frictional forces. The surface error depends on the maximum gradient and is shown to be negligible in certain special cases. The effect of the "spin" terms is shown to be dependent on the scale factor. Frictional losses are the most important, and a method of measuring these in a special set up is described. The author concludes that the model should be as small as possible and that the error in the ball's kinetic energy (due to friction) can be kept less than 2 per cent of the maximum potential differences between points of the boundary.

In connection with the discussion on this paper the electrolytic tank of BOOTHROYD, CHERRY and MAKAR [cf. *MTAC*, v. 33, p. 49–50] and the resistance networks of E. E. HUTCHINGS and of G. LIEBMANN [cf. *MTAC*, v. 35, p. 179] were demonstrated. The accuracies of the various systems were compared also in these discussions.

F. J. M.

## NOTES

129. ZEROS OF $I_{n+1}(x)J_n(x) + J_{n+1}(x)I_n(x)$. A table of the first ten zeros of $f_n(x) \equiv I_{n+1}(x)J_n(x) + J_{n+1}(x)I_n(x)$ for $n = 0$, 1, 2, and 3, was published by AIREY[1]. This table is extended herewith to include all zeros $\leq 20$. For the sake of completeness, Airey's values are reproduced here, with the kind permission of the editors of the *Proceedings*.

Airey's values were compared with those of CARRINGTON,[2] who gave all zeros $\leq 16$. Corresponding to $n = 0$, Airey gave the first zero as 3.1955, whereas Carrington gave 3.1961. This entry was recomputed; the true value to five decimals is 3.19622. Other entries in Airey's differ from Carrington's by at most a unit in the third decimal place, where both authors give the same zeros. Differences of Airey's values show no obvious errors, but his entries were not otherwise verified by us.

G. FRANKE[3] published the first two zeros for $n = 4$ and the first zero for $n = 5$, 6, and 7, to one, two, or three decimals. Comparison of his entries with those published here shows that his last place is not correct.

The entries given here were computed by inverse interpolation in $[\exp(-x)]f_n(x)$, with the aid of values of $I_n(x)$ which were made available to us in manuscript form by J. C. P. MILLER. The extensive tables of $J_n(x)$ of the Harvard Computation Laboratory provided the other required tabular values. GLADYS FRANKLIN of the NBSINA performed the computations. Entries for $n > 3$ are correct to within $\pm .00002$. The work was carried out with the aid of funds provided by the ONR, in connection with an eigenvalue problem investigated by N. ARONSZAJN.

Zeros of $f_n(x)$

| $s$ | $n = 0$ | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ |
|---|---|---|---|---|---|
| 1 | 3.19622 | 4.611 | 5.906 | 7.144 | 8.34661 |
| 2 | 6.3064 | 7.799 | 9.197 | 10.536 | 11.83672 |
| 3 | 9.4395 | 10.958 | 12.402 | 13.795 | 15.14987 |
| 4 | 12.5771 | 14.109 | 15.579 | 17.005 | 18.39596 |
| 5 | 15.7164 | 17.256 | 18.745 | 20.192 | |
| 6 | 18.8565 | 20.401 | 21.901 | 23.366 | |
| 7 | 21.9971 | 23.545 | 25.055 | 26.532 | |
| 8 | 25.1379 | 26.689 | 28.205 | 29.693 | |
| 9 | 28.2790 | 29.832 | 31.354 | 32.849 | |
| 10 | 31.4200 | 32.975 | 34.502 | 36.003 | |

| $s$ | $n = 5$ | $n = 6$ | $n = 7$ | $n = 8$ | $n = 9$ |
|---|---|---|---|---|---|
| 1 | 9.52570 | 10.68703 | 11.83453 | 12.97091 | 14.09809 |
| 2 | 13.10736 | 14.35516 | 15.58455 | 16.79874 | 18.00010 |
| 3 | 16.47508 | 17.77643 | 19.05806 | | |
| 4 | 19.75828 | | | | |

| $s$ | $n = 10$ | $n = 11$ | $n = 12$ | $n = 13$ | $n = 14$ |
|---|---|---|---|---|---|
| 1 | 15.21753 | 16.33031 | 17.43732 | 18.53925 | 19.63669 |
| 2 | 19.19045 | | | | |

G. Blanch

NBSINA

[1] J. R. Airey, "The vibrations of circular plates and their relation to Bessel functions." Phys. Soc., London, *Proc.*, v. 23, Dec. 1910–Aug. 1911, p. 225–232.

[2] H. Carrington, "The frequencies of vibration of flat circular plates fixed at the circumference," *Phil. Mag.*, s. 6, v. 50, 1925, p. 1261–1264.

[3] Georg Franke, "Erzwungene Schwingungen einer eigenspannten kreisförmigen Platte." *Annalen der Physik*, s. 5, v. 2, 1929, p. 649–675.

**130.** A Method of Factorisation Using a High-Speed Computer. The usual process of finding the factors or establishing the primality of a large number $N$ involves the determination of the remainders $r_n$ in the equation

$$N = f_n q_n + r_n,\ 0 \leqq r_n < f_n. \tag{1}$$

Only prime numbers need be taken for $f_n$, but in practice we take all integers less than $N^{1/2}$ except multiples of small primes 2, 3, 5, etc.

If $N$ is a large prime the work involved is considerable and the time for the complete operation depends mainly on the speed of division of the computer.

This note describes a method of factorisation which replaces the division routine at least for the range of $f_n$ from $2N^{1/3}$ to $N^{1/2}$ by operations which are more rapid on some machines. Since for large $N$ this range includes most of the possible $f_n$ a considerable saving of time is effected on these machines.

The method uses the known relation between the selected $f_n$ to determine from equation (1) a relation between successive $r_n$ from which $r_n$ can be obtained by a few additions and subtractions and duplications.

Suppose, for simplicity, that $f_{n+1} = f_n + 2$. Then if $\nabla$ denotes the backward difference operator we can easily show that

$$\nabla^2 r_{n+1} = -4\nabla q_n - (f_n + 2)\nabla^2 q_{n+1}. \tag{2}$$

Now

$$(N/f_n) - 1 < q_n \leqq (N/f_n),$$

so that

$$\nabla^2(N/f_n) - 2 < \nabla^2 q_n < \nabla^2(N/f_n) + 2. \tag{3}$$

also

$$\nabla^2(N/f_n) = 16/\{f_n(f_n - 2)(f_n - 4)\}. \tag{4}$$

For $f_n \geqq 2N^{1/2} + 1$, we have from (4)

$$\nabla^2(N/f_{n+1}) = \frac{8N}{(2N^{1/2} + 3)(2N^{1/2} + 1)(2N^{1/2} - 1)} < 1.$$

It follows from (3) that $\nabla^2 q_{n+1}$ can have only the values $-1, 0, 1$ or $2$.

The process of testing whether $f_{n+1}$ is a factor of $N$ then consists of the following steps, starting from the point at which, for some $n$, the quantities $r_{n-1}, r_n, f_n, \nabla q_n$ are stored in memory positions.

A Replace $f_n$ by $f_{n+1} = f_n + 2$.
B Replace $r_n$ by $r_{n+1} = 2r_n - r_{n-1} - 4\nabla q_n - (f_n + 2)X$, where $X$ is one of $-1, 0, 1, 2$, to be chosen uniquely so that $0 \leq r_{n+1} < f_n + 2$.
C Replace $r_{n-1}$ by $r_n$.
D Replace $\nabla q_n$ by $\nabla q_{n+1} = \nabla q_n + X$.
E Test for $r_{n+1} = 0$.
F Test for $f_{n+1} \geqq N^{1/2}$.

This process is manifestly simple. On the pilot model of the ACE we have used it to establish the primality or discover a factor of a twelve decimal digit number in less than 15 minutes, each step in the above process taking 1 millisecond.

For $f_n < 2N^{1/2} + 1$ some of the advantage of the process is lost since $X$ is no longer necessarily $-1, 0, 1$ or $2$ but has to be found at step (B) above by dividing $2r_n - r_n - 1 - 4\nabla q_n$ by $f_n + 2$. By using this process throughout, however, a powerful check is provided. Each $r_n$ depends on the previous remainders so that if the machine finds no factor and if the last remainder found is correct, as can be verified by direct division by the last trial factor, considerable confidence can be placed in the result that the particular number is prime.

Modifications of the above process make possible the treatment of cases in which $f_n$ is an arithmetical progression such as $4n + 1$.

The work described above has been carried out as part of the research program of the National Physical Laboratory and this article is published by permission of the Director of the Laboratory.

G. G. Alway

National Physical Laboratory
Teddington, Middlesex
England

[Editorial Note: Roselyn Lipkis reports that, as applied to the SWAC, the above method speeds up the factoring process by a factor of 8. All six steps A–F are performed in approximately 1.3 milliseconds.]

**131.** RECENT DISCOVERIES OF LARGE PRIMES. Ever since LUCAS announced the discovery of the prime $2^{127} - 1$ in 1876, many attempts have been made to discover larger primes. These attempts have succeeded only recently as follows:

(a) A. FERRIER[1] has identified $(2^{148} + 1)/17$ as a prime, using a method based on the converse of Fermat's theorem and a desk calculator.
(b) Using the same method and the EDSAC, WHEELER and MILLER[2,3] have proved the primality of $1 + k(2^{127} - 1)$ for $k = 114, 124, 388, 408, 498, 696, 738, 744, 780, 934, 978$, and finally $1 + 180(2^{127} - 1)^2$, a number of 79 decimal digits.
(c) Using the standard LUCAS test for Mersenne primes as programmed by R. M. ROBINSON, the SWAC has discovered the primes $2^{521} - 1$ and $2^{607} - 1$ on January 30, 1952. These lead to the 13th and 14th perfect numbers.

D. H. L.

[1] Letter of July 14, 1951.
[2] J. C. P. MILLER & D. J. WHEELER, "Large prime numbers," *Nature*, v. 168, 1951, p. 838.
[3] J. C. P. MILLER, "Large primes," *Eureka*, 1951, no. 14, p. 10–11.

## QUERIES

**40.** TABLE OF MULTIPLICATION.—Brown University has just acquired a copy of J. B. Oyon, *Tables de Multiplication, A l'Usage de MM. les Géomètres*. Second edition, Paris, 1812; quarto, 507 p., bound in two volumes. This work gives the product of all integer pairs up to $509 \times 500$. There is no indication of any author's name, but in the *Catalogue Général des Livres emprimés de la Bibliothèque National*, v. 128, we find the work listed under Oyon's name in a third edition, Paris, 1824; and also a fourth edition, v. 2, Lyon, 1864, which seems to continue the table to $509 \times 1000$. The Catalogue's first publication listed after Oyon's name is a 4-volume *Collection des Lois, Arrêtés, Instructions* . . . , Paris, 1804–1808.

Where may information concerning Oyon be found? When was the first edition of his *Tables* published and where may it be consulted? The third edition is also in the British Museum. What other libraries have the second and fourth editions?

R. C. ARCHIBALD

Brown University
Providence, R. I.

## CORRIGENDA

V. 5, p. 67, eqn. (2), *for* = *read* $\doteq$.
V. 5, p. 116, l. −10, −9, *for* Column of Probabilities *read* Column of Expectations.
V. 5, p. 118, l. 20, *for* ten *read* $n + 2$.
V. 5, p. 119, l. −8, *for* C to N *read* C to N$\acute{X}$.
V. 5, p. 130, l. 17, *for* $k_2/h$ *read* $k_2/2$.
V. 5, p. 163, l. 11, *for* $K$ *read* $\bar{K}$.
V. 5, p. 167, l. −15, *for* A. C. *read* C. A.
V. 5, p. 167, l. −14, *for* Camb. Phil. Soc. *Proc.*, *read* *Phil. Mag.*
V. 5, p. 258, l. −3, *for* **49** *read* **39**.

A.

B.

C.

D.

E.

F.

G.

H.

I.

J.

K.

L.

M.

N.

O.

P.

Q.

R.

S.

T.

U.

V.

Z.

# CLASSIFICATION OF TABLES

A. Arithmetical Tables. Mathematical Constants

B. Powers

C. Logarithms

D. Circular Functions

E. Hyperbolic and Exponential Functions

F. Theory of Numbers

G. Higher Algebra

H. Numerical Solution of Equations

I. Finite Differences. Interpolation

J. Summation of Series

K. Statistics

L. Higher Mathematical Functions

M. Integrals

N. Interest and Investment

O. Actuarial Science

P. Engineering

Q. Astronomy

R. Geodesy

S. Physics, Geophysics, Crystallography

T. Chemistry

U. Navigation

V. Aerodynamics, Hydrodynamics, Ballistics

Z. Calculating Machines and Mechanical Computation

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